Proxy Advisory Firms: The Economics of Selling Information to Voters*

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Abstract

Proxy advisors play an important role by providing investors with research and recommendations on how to vote their shares. This paper examines how proxy advisors affect the quality of corporate decision-making. We analyze a model in which a monopolistic advisor offers to sell information to shareholders, who decide whether to acquire private information and/or buy the advisor's recommendation, and how to cast their votes. We show that the proxy advisor's presence can decrease the quality of decision-making, even if its information is more precise than shareholders' information and no party has a conflict of interest. This is because there is a wedge between privately optimal and socially optimal information acquisition decisions, leading to inefficient crowding out of private information production. We also evaluate several existing proposals on regulating proxy advisors and show that some suggested policies, such as reducing proxy advisors' market power or increasing the transparency of their methodologies, can have a negative effect.

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1 Introduction

Proxy advisory firms provide shareholders with research and recommendations on how to cast their votes at shareholder meetings of public companies. For highly diversified institutional investors, the costs of performing independent research on each issue on the agenda in each of their portfolio companies are substantial. The institution may prefer to pay a fee and buy information from a proxy advisory firm instead. A shareholder subscribing to proxy advisory services receives a report that contains recommendations on all management and shareholder proposals to be voted on, as well as the analysis underlying these recommendations. The largest proxy advisor, Institutional Shareholder Services (ISS), has over 1,600 institutional clients and covers almost 40,000 meetings around the world.

In the last years, the demand for proxy advisory services has substantially increased due to several factors – the rise in institutional ownership, the 2003 SEC rule requiring mutual funds to vote in their clients' best interests, and the increased volume and complexity of issues voted upon, which was brought by the introduction of mandatory say-on-pay and the growing number of proxy contests and shareholder proposals. By now, there is strong empirical evidence that proxy advisors' recommendations have a large influence on voting outcomes.[1] This influence has attracted the attention of the SEC and regulatory bodies in other countries and has led to a number of policy proposals seeking to increase the transparency of the proxy advisory industry, make it more competitive, and reduce potential conflicts of interest.

While proxy advisory firms have a strong influence on shareholder votes, the costs and benefits of this influence are not well understood. The goal of this paper is to theoretically examine how proxy advisors affect the quality of corporate decision-making and to analyze the effects of the suggested policy proposals. We show that although proxy advisors provide additional valuable information to investors, their presence can lead to less informed corporate decisions, even if their recommendations are completely unbiased and more informative than the research shareholders could do on their own. As a result, some frequently suggested policies, such as reducing proxy advisors' market power or increasing disclosure about their methodologies and conflicts of interest, can actually have a negative effect.

[1] See Alexander et al. (2010), Ertimur, Ferri, and Oesch (2013), Iliev and Lowry (2015), Larcker, McCall, and Ormazabal (2015), and Malenko and Shen (2016), among others.

We develop a tractable model of shareholder voting in the presence of a proxy advisory firm. Shareholders are voting on a proposal that can increase or decrease firm value with equal probability. Each shareholder can acquire information about the value of the proposal from two sources – do his own independent research or buy information from the proxy advisor. For example, in practice, some institutions have their own proxy research departments, while others strongly rely on proxy advisors' recommendations.[2] More specifically, there is a monopolistic proxy advisor that has an informative signal about the proposal. The proxy advisor sets a fee that maximizes its expected profits and offers to sell its signal to the shareholders for this fee. Each shareholder then independently decides whether to buy the proxy advisor's signal, to pay a cost to acquire his own signal, to acquire both signals, or to remain uninformed. After observing the signals he acquired, each shareholder decides how to vote, and the proposal is implemented if it is approved by the majority of shareholders.

Our main result is that in this setup, unless the proxy advisor's signal is sufficiently precise, its presence decreases firm value: shareholder votes would be more informed if the proxy advisory firm did not exist. This is true even though the proxy advisor's recommendations are unbiased, shareholders have no conflicts of interest, and even if the proxy advisor's signal is more precise than each shareholder's independent signal. The intuition for this result comes from a combination of two effects: the difference between the privately optimal and socially optimal information acquisition decisions by shareholders, and strategic fee setting by the proxy advisor.

To see the intuition in the simplest way, consider the following example. Suppose that the fee set by the proxy advisor equals the cost of private information acquisition and that all shareholders except one are either uninformed or are following the proxy advisor. Consider the remaining shareholder's choice between acquiring his own private information and buying the proxy advisor's recommendation. This choice only affects the shareholder's utility when the vote turns out to be close, i.e., the votes of other shareholders are split equally. Conditional on this event, the shareholder does not infer any additional information about the informativeness of the proxy advisor's recommendation. Hence, the shareholder's optimal choice between acquiring the private vs. the proxy advisor's signal depends entirely on which of the two signals is a priori more precise and does not depend on how many

[2] See the Government Accountability Office report on proxy advisors (GAO, 2007) and the WSJ article "For Proxy Advisers, Influence Wanes," May 22, 2013. Iliev and Lowry (2015) show that there is significant heterogeneity among institutions in the extent to which they rely on ISS.

other shareholders already follow the proxy advisor. In particular, the shareholder finds it privately optimal to acquire the proxy advisor's signal as long as it is more precise than the private signal, even if it is only marginally more precise. This, however, is socially inefficient: the voting outcome will be more efficient if many shareholders follow their private signals than if all of them follow the proxy advisor, unless the proxy advisor's information is very precise. This is because when shareholders follow their private signals, they make mistakes that are independent (or, more generally, imperfectly correlated) conditional on the state. In contrast, when all shareholders follow the same, albeit a more precise signal, their mistakes are perfectly correlated, which can increase the probability of an incorrect decision, i.e., of shareholders approving a proposal that should be rejected or vice versa.

More generally, a shareholder who acquires information (privately or from the proxy advisor) imposes a positive externality on other shareholders. When some other shareholders already follow the proxy advisor, this externality is higher if the shareholder acquires information privately than if he buys the proxy advisor's recommendation. As a result, because of the collective action problem, the proxy advisor's presence crowds out private information acquisition to a greater extent than would be socially optimal, which can increase the probability of an incorrect decision being made. Overall, we show that the presence of the proxy advisor has a negative effect on firm value if the precision of its signal is not too high, but has a positive effect if its signal is precise enough.

The fact that the proxy advisor sets its fee strategically, aiming to maximize its own profits rather than firm value, exacerbates its negative influence when its signal is not too precise and decreases its potential positive influence when its information is sufficiently precise. Intuitively, when the proxy advisor's information is not too precise, firm value would be maximized if its recommendations could be made prohibitively costly to deter the shareholders from buying them all together. Similarly, when the proxy advisor's information is sufficiently precise, firm value would be maximized if the price of its recommendations could be made as low as possible, at the level that just compensated the proxy advisor for the cost of producing information. Clearly, neither of these policies corresponds to what the monopolistic proxy advisor finds optimal to do. Interestingly, strategic fee setting by the proxy advisor implies that its presence decreases firm value even if its information is perfectly precise, as long as the quality of decision-making without the advisor is sufficiently high. Intuitively, to maximize profits, the proxy advisor chooses to sell its perfectly precise recommendation only to a fraction of investors. Together with crowding out of private information acquisition, this implies that a large fraction of shareholders vote uninformatively, decreasing the quality of corporate decisions.

We use the model to evaluate the costs and benefits of several policy proposals that have been put forward by regulators, investors, and other market participants to regulate proxy advisors.[3] Some of these proposals aim to increase the transparency of the proxy advisory industry. They include requiring proxy advisors to disclose the methodologies, assumptions, and data supporting their recommendations, disclose any conflicts of interest they may have, and even to make their recommendations public. Other proposals aim to reduce the market power of proxy advisors. Currently, the industry is very concentrated: ISS controls 61% of the market and has more clients than all of the other proxy advisors combined, and the second largest proxy advisor, Glass Lewis, controls 36% of the market. As a result, market participants have been pushing for reducing the two proxy advisors' market power in order to lower the costs of proxy advisory services (GAO, 2007).

Interestingly, our results suggest that decreasing the proxy advisor's market power and lowering its fees is not always beneficial: whether this leads to more informed voting decisions depends on the quality of the advisor's information. To see this, suppose that the proxy advisor's information is not too precise, so that there is inefficient overreliance on its recommendations, but some private information acquisition still occurs. In this case, lowering the proxy advisor's fees would encourage even more investors to buy its recommendations instead of acquiring private information, which would be detrimental for the quality of decision-making. On the other hand, if the proxy advisor's information is sufficiently precise, reducing its fees and thereby encouraging more shareholders to buy its recommendations would be beneficial. Similarly, we show that improving the disclosure of the proxy advisor's methodologies and conflicts of interest, which we model as increasing the transparency about the quality of its recommendations, can have both a positive and negative effect, depending on the precision of its information relative to that of shareholders. Overall, our results suggest that any regulation of proxy advisors should carefully take into account how it will affect private information acquisition by investors and how informative proxy advisors' recommendations are.

Our model deliberately abstracts from a common critique of proxy advisors – potential conflicts of interest due to ISS's consulting relationships with corporations and Glass Lewis'

[3] See Edelman (2013) and the October 20, 2010 Shareholder Communications Coalition Letter to the SEC for detailed discussions of these proposals.

ownership by one of Canada's largest pension funds. Although the potential for conflicts of interest should be an important consideration, our goal is to emphasize that even without conflicts of interest,[4] the presence of proxy advisors may lead to less efficient voting outcomes by crowding out private information production by institutions. Of course, the extent to which the concern we emphasize is important depends on two factors. First, if proxy advisors' information is sufficiently precise, their presence is beneficial despite the crowding out effect. Second, even if their information is not too precise, their presence is nevertheless beneficial if no crowding out occurs, i.e., if institutions would not acquire private information even if proxy advisors did not exist.[5]

Our paper contributes to the literature on the sale of information. It includes the literature on selling information to traders in financial markets (e.g., Admati and Pfleiderer, 1986, 1990; Fishman and Hagerty, 1995; Cespa, 2008; and Garcia and Sangiorgi, 2011, among others), as well as information sales in other contexts (e.g., Bergemann, Bonatti, and Smolin, 2015). To our knowledge, our paper is the first to study the sale of information to agents who can also engage in private information acquisition. Our second contribution is to examine information sales in a strategic voting context. There are two important differences between selling information to voters and financial traders, which make our setting and results different from those in the literature: first, voters have common interests while traders compete with each other; second, in voting, voters mostly care about the event when their vote makes a difference.

Our paper is also related to the literature on strategic voting, which studies how information that is dispersed among voters is aggregated in the vote (e.g., Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1998). It is mostly related to papers that analyze endogenous information acquisition by voters (Persico, 2004; Martinelli, 2006; Gerardi and Yariv, 2008; Gershkov and Szentes, 2009; Khanna and Schroder, 2015). Differently from these papers, which focus on how the number of voters and the decision-making rule affect information acquisition and the quality of decision-making, our focus is on the effect of information sales by a third party. Alonso and Camara (2016), Chakraborty and Harbaugh (2010), Jackson and Tan (2013), and Schnakenberg (2015) analyze information provision by

[4]The SEC's 2014 Staff Legal Bulletin No. 20 tried to address the potential conflict of interest concerns by requiring that proxy advisors disclose any conflicts of interest to their clients, and that institutions carefully consider such conflicts of interest in their decisions on whether to follow proxy advisors' recommendations.

[5]Note, however, that given the SEC's 2003 rule, an institution that did not take any effort to make an informed voting decision could be exposed to legal risk for violating its fiduciary duties to its clients.

biased senders to voters, in the form of either communication or Bayesian persuasion. Their focus is on how the sender exploits heterogeneity in voters' preferences to sway the outcome in his favor, while our model features no conflicts of interest between parties and instead focuses on the sale of information and crowding out of private information acquisition.

In the corporate finance context, voting has been analyzed by Maug (1999) and Maug and Yilmaz (2002), who examine conflicts of interest between voters; Bond and Eraslan (2010), who study voting on an endogenous agenda in the debt restructuring context (among other contexts); Brav and Mathews (2011), who analyze empty voting; and Levit and Malenko (2011), who study nonbinding shareholder voting. Our paper contributes to this literature by analyzing another important institutional feature of corporate voting – the presence of proxy advisors. Li (2014) provides evidence that Glass Lewis's entry into the industry alleviated ISS's bias towards management and also builds a model studying proxy advisors' conflicts of interest, but his model does not feature strategic voting and privately informed shareholders.

Finally, our paper is related to the large literature on the interplay between public and private information.[6] The closest papers to ours examine how public information disclosure affects investors' incentives for private information production (e.g., Diamond, 1985; Boot and Thakor, 2001). Differently from these papers, where the interplay between public information and private information acquisition works through trading profit considerations, the mechanism in our paper is through shareholders' beliefs about the effect of their decisions on voting outcomes. Another difference is that we focus on the sale of information by a profit-maximizing seller, rather than on free distribution of information by the firm. Another related paper is Hellwig and Veldkamp (2009), who study information acquisition decisions in a beauty-contest setup and show that when agents' actions are strategic complements (substitutes), the value of additional information increases (decreases) when other agents become more informed.

The remainder of the paper is organized as follows. Section 2 describes the setup and solves for the benchmark case of shareholder voting without a proxy advisor. Section 3 analyzes shareholders' information acquisition and voting decisions in the presence of a proxy advisor and derives implications for the efficiency of decision-making. Section 4 discusses the

[6] For example, the literature started by Morris and Shin (2002) studies the use of public vs. private information by agents who have coordination motives, and Bond and Goldstein (2015) analyze how public disclosure affects investors' incentives to trade on private information and the resulting feedback effects.

optimal pricing strategy of a monopolistic proxy advisor, and Section 5 analyzes the effects of several policy proposals. Finally, Section 6 concludes.

2 Model setup

We adopt the standard setup in the strategic voting literature (e.g., Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1998) and augment it by introducing an advisor that offers to sell its signal to the voters.

The firm is owned by $N \geq 3$ shareholders, where N is odd. Each shareholder owns the same stake in the firm (for simplicity, one share), and each share provides one vote. It is easiest to think about these shareholders as the company's institutional investors: given their often significant holdings in the companies and their fiduciary duties to their clients, they are likely to have incentives to vote in an informed way and hence to incur the costs of private information acquisition or the costs of buying proxy advisors' recommendations.

There is a proposal to be voted on at the shareholder meeting, which is implemented if it is approved by the majority, i.e., if at least $\frac{N+1}{2}$ shareholders vote for it.[7] Let d denote whether the proposal is accepted ($d = 1$) or rejected ($d = 0$). The value of the proposal, and thus the optimal decision $d^* \in \{0, 1\}$, depends on the unknown state $\theta \in \{0, 1\}$, where both states are equally likely. Without loss of generality, assume that the efficient decision is to match the state, i.e., accept the proposal if $\theta = 1$ and reject it if $\theta = 0$. Specifically, firm value per share increases by one if the proposal is accepted in state $\theta = 1$ and decreases by one if it is accepted in state $\theta = 0$. If the proposal is rejected, firm value does not change. Denoting the change in firm value per share by $u(d, \theta)$,

$$\begin{aligned} u(1,\theta) &= \begin{cases} 1, & \text{if } \theta = 1, \\ -1, & \text{if } \theta = 0, \end{cases} \\ u(0,\theta) &= 0. \end{aligned} \tag{1}$$

Shareholders maximize the value of their shares minus any costs of information acquisi-

[7] While this formulation assumes that the vote is binding, our setup can also apply to nonbinding votes. First, the 50% voting threshold is an important cutoff, passing which leads to a significantly higher probability of proposal implementation even if the vote is nonbinding (e.g., Ertimur, Ferri, and Stubben, 2010; Cuñat, Gine, and Guadalupe, 2012). Second, Levit and Malenko (2011) show that nonbinding voting is equivalent to binding voting with an endogenously determined voting cutoff that depends on company and proposal characteristics.

tion. Each shareholder can potentially get access to two signals – his private signal and the recommendation of an advisor (the proxy advisory firm). Specifically, the advisor's information is represented by signal ("recommendation") $r \in \{0,1\}$, whose precision is given by $\pi \in [\frac{1}{2}, 1]$:[8]

$$\Pr(r = 1|\theta = 1) = \Pr(r = 0|\theta = 0) = \pi. \tag{2}$$

Each shareholder can buy the advisor's recommendation for fee f, which is optimally set by the advisor at the initial stage. We assume that the advisor's recommendation is simply given by r, so that a shareholder who subscribes to the advisor's services observes r.[9]

In addition to the advisor's signal, each shareholder has access to a private information acquisition technology, whereby shareholder i can acquire a private signal $s_i \in \{0,1\}$ at a cost $c > 0$. The precision of the private signal is given by $p \in [\frac{1}{2}, 1]$:

$$\Pr(s_i = 1|\theta = 1) = \Pr(s_i = 0|\theta = 0) = p. \tag{3}$$

All signals are independent conditional on state θ.

The timing of the model is illustrated in Figure 1. There are four stages. At Stage 1, the advisor sets fee f that it charges each shareholder for the recommendation. At Stage 2, each shareholder independently and simultaneously decides on whether to acquire his private signal at cost c, acquire the recommendation from the advisor for fee f, acquire both signals, or remain uninformed. At Stage 3, each shareholder i privately observes the signals he acquired, if any, and decides how to vote to maximize the expected firm value: $v_i \in \{0,1\}$, where $v_i = 1$ ($v_i = 0$) corresponds to voting in favor of (against) the proposal. The votes are cast simultaneously. At Stage 4, the proposal is implemented or not, depending on whether the majority of shareholders voted for it, and the payoffs are realized.

We focus on symmetric Bayes-Nash equilibria. Symmetry means two things. First, all shareholders follow the same information acquisition strategy, and at the voting stage, all shareholders of one type (i.e., those who acquired the recommendation from the advisor; those who acquired a private signal; those who acquired neither; and those who acquired

[8] For example, in the context of proxy contests, Alexander et al. (2010) find that ISS recommendations in favor of dissidents are accompanied by a 3.8% abnormal return and seem to convey information not only about the likelihood of dissident victory, but also about the dissident's value to the firm.

[9] In practice, proxy advisors sometimes give personalized vote recommendations to clients that have a strong position on particular issues, e.g., on corporate social responsibility proposals. Such behavior would arise in our model if we assumed that shareholders have heterogeneous preferences, the feature that we abstract from in this paper.

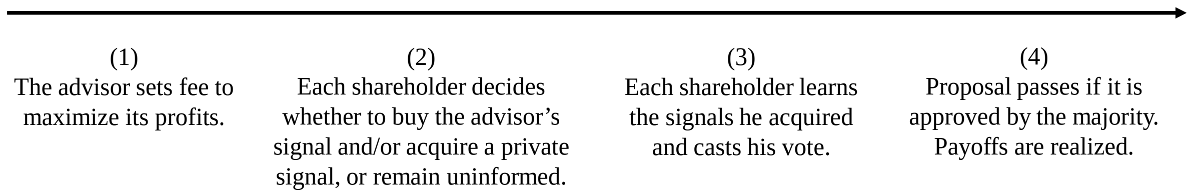

Figure 1. Timeline of the model.

both) use the same voting strategy, denoted $w_r(r) : \{0,1\} \to [0,1]$, $w_s(s_i) : \{0,1\} \to [0,1]$, $w_0 \in [0,1]$, and $w_{rs}(r, s_i) : \{0,1\} \times \{0,1\} \to [0,1]$, where $w_r(\cdot)$, $w_s(\cdot)$, w_0, and $w_{rs}(\cdot)$ denote the probability of voting "for" given the respective information set. Second, since the model is fully symmetric in states and signals, we look for equilibria that are symmetric around the state: $w_s(s_i) = 1 - w_s(1 - s_i)$, $w_r(r) = 1 - w_r(1 - r)$, $w_0 = \frac{1}{2}$, and $w_{rs}(r, s_i) = 1 - w_{rs}(1 - r, 1 - s_i)$ $\forall s_i \in \{0,1\}$ and $\forall r \in \{0,1\}$.[10] In what follows, we refer to symmetric equilibria as simply equilibria.

We assume that shareholders cannot abstain from voting on the proposal. This assumption matches reality: in practice, institutional investors rarely abstain from voting, probably because of the fear of violating their fiduciary duties or of being perceived as uninformed. For example, according to our calculations based on the ISS Voting Analytics database for 2003-2012, mutual funds do not vote or formally abstain in less than 1% of cases.[11]

2.1 Benchmark: Voting without the proxy advisory firm

As a benchmark, it is useful to consider shareholder voting in the absence of the advisor. In this case, the model is an extension of the standard problem of strategic voting, augmented

[10] The symmetry assumption allows us to eliminate "uninformative" equilibria, where all shareholders remain uninformed and then always vote in the same direction. Since a shareholder's vote is never pivotal, remaining uninformed is optimal.

[11] Moreover, the equilibrium of our model will also be an equilibrium if we extend the model by allowing each shareholder to abstain from voting and assume that in the event of a tie, the proposal is implemented randomly. Consider an uninformed shareholder and note that his vote only matters if the votes of other shareholders are split equally. Conditional on this event, both states are equally likely and hence the shareholder is indifferent between it being accepted or rejected. If the shareholder abstains from voting, the proposal is implemented randomly, uncorrelated with the state; if the shareholder does not abstain from voting, he randomizes between voting for and against and hence the implementation of the proposal is also independent of the state. Hence, the uninformed shareholder is indifferent between abstaining and not abstaining, and thus our equilibrium indeed continues to exist in this extended model.

by the information acquisition stage.[12] A variation of this problem has been studied by, e.g., Persico (2004), but differently from us, he focuses on equilibria where all voters get informed with probability one.

An equilibrium is given by probability $q \in [0,1]$ with which each shareholder acquires a private signal; function $w_s(s)$, the probability of voting "for" given signal s; and probability $w_0 = \frac{1}{2}$ of voting "for" given no information.

Note that in equilibrium, a shareholder who acquires a private signal must follow it. Indeed, if the shareholder always votes in the same way regardless of his signal, he is better off not paying for the signal in the first place. Similarly, if the shareholder mixes (and hence is indifferent) between voting according to his signal and against it for at least one realization of the signal, then his utility would not change if he voted in the same way regardless of his signal, so he is again better off not acquiring the signal. Thus, the only possible voting equilibrium where information has positive value is the one where each informed shareholder votes according to his signal. The proof of Proposition 1 shows that such an equilibrium exists for any q.

Given the equilibrium at the voting stage, we can solve for the equilibrium at the information acquisition stage. Consider shareholder i contemplating whether to acquire a private signal, given that he expects each other shareholder to acquire his private signal with probability q. Conditional on the shareholder's private signal being $s_i = 1$, whether he is informed or not only makes a difference if the number of "for" votes among other shareholders is exactly $\frac{N-1}{2}$. Let us denote this set of events by PIV_i. In this case, by acquiring the signal, the shareholder votes "for" for sure, instead of randomizing between voting "for" and "against," so his utility from being informed is $\frac{1}{2}\mathbb{E}\left[u(1,\theta)|s_i = 1, PIV_i\right]$. Similarly, conditional on his private signal being $s_i = 0$, the shareholder's utility from being informed is $-\frac{1}{2}\mathbb{E}\left[u(1,\theta)|s_i = 0, PIV_i\right]$. Overall, the shareholder's value of acquiring a private signal is

$$\begin{aligned} V(q) = &\Pr(s_i = 1)\Pr(PIV_i|s_i = 1)\tfrac{1}{2}\mathbb{E}\left[u(1,\theta)|s_i = 1, PIV_i\right] \\ &- \Pr(s_i = 0)\Pr(PIV_i|s_i = 0)\tfrac{1}{2}\mathbb{E}\left[u(1,\theta)|s_i = 0, PIV_i\right]. \end{aligned}$$

It is useful to define function $P(x,n,k)$ as the probability that the proposal gets k votes

[12] Maug and Rydqvist (2009) provide evidence consistent with shareholders voting strategically.

out of n when each shareholder independently votes for the proposal with probability x:

$$P(x,n,k) \equiv C_n^k x^k (1-x)^{n-k}, \tag{4}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Using the symmetry of the setup and Bayes' rule, we can write $V(q)$ as (see the proof of Proposition 1 for the derivation):

$$V(q) = (p-\frac{1}{2})P(qp+(1-q)\frac{1}{2}, N-1, \frac{N-1}{2}) = (p-\frac{1}{2})C_{N-1}^{\frac{N-1}{2}}\left(\frac{1}{4}-q^2(p-\frac{1}{2})^2\right)^{\frac{N-1}{2}} \tag{5}$$

The intuition behind (5) is simple. Consider one shareholder. When any other shareholder acquires his private signal with probability q, the probability that he votes correctly is $qp+(1-q)\frac{1}{2}$: the probability of a correct vote equals the precision of the signal p if the shareholder gets informed, and equals $\frac{1}{2}$ if he does not. Thus, the shareholder's vote determines the decision with probability $P\left(qp+(1-q)\frac{1}{2}, N-1, \frac{N-1}{2}\right)$. Conditional on this event, the value of the signal to the shareholder equals $p-\frac{1}{2}$, implying that the expected value from getting informed is (5). The value of information $V(q)$ is decreasing in the number of shareholders N or, equivalently, increasing in the stake of each shareholder. This is because with more shareholders, the shareholder's vote is less likely to make a difference in the final decision, reducing his incentives to acquire information. In addition, $V(q)$ is decreasing in the probability q with which other shareholders acquire their private signals. Intuitively, as other shareholders become more informed, they are more likely to vote in the same way, which reduces the chances of a close vote when the shareholder's information becomes critical.

In deciding whether to acquire the private signal, shareholder i compares the expected value from the signal $V(q)$ with cost c and acquires the signal if and only if $V(q) \geq c$. Since the value of shareholder i's information is strictly decreasing in the expected fraction q of other shareholders acquiring information, the equilibrium probability with which each shareholder gets informed is determined as a unique solution to $V(q) = c$, unless c is very low or very high. If c is very low or very high, then either all shareholders acquire information or none of them do. This equilibrium is summarized in Proposition 1 below.

Proposition 1 (equilibrium without the advisor). *There exists a unique symmetric* *equilibrium. Each shareholder acquires a private signal with probability q^*, given by*

$$q^* = \begin{cases} 1, & \text{if } c \leq \underline{c} \equiv V(1) = \left(p - \frac{1}{2}\right) C_{N-1}^{\frac{N-1}{2}} \left(\frac{1}{4} - \left(p - \frac{1}{2}\right)^2\right)^{\frac{N-1}{2}}, \\ q_0^* \equiv \frac{2}{2p-1}\Lambda, & \text{if } c \in (\underline{c}, \bar{c}), \\ 0, & \text{if } c \geq \bar{c} \equiv V(0) = \left(p - \frac{1}{2}\right) C_{N-1}^{\frac{N-1}{2}} 2^{1-N}. \end{cases} \tag{6}$$

where $\Lambda \equiv \sqrt{\frac{1}{4} - \left(\frac{c}{p-\frac{1}{2}} \frac{1}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}$. *At the voting stage, a shareholder with signal s_i votes "for" ($v_i = 1$) if $s_i = 1$ and "against" ($v_i = 0$) if $s_i = 0$, and an uninformed shareholder votes "for" with probability* 0.5.

In what follows, we assume that $c \in (\underline{c}, \bar{c})$, that is, the interior solution occurs in the model without the advisor.

Assumption 1. *$c \in (\underline{c}, \bar{c})$, so that $q^* \in (0, 1)$ in the model without the advisor.*

The rationale for Assumption 1 is simple: we want to focus on the cases where private information acquisition is a relevant margin. If $c > \bar{c}$, then the problem becomes trivial: private information acquisition is irrelevant. In this case, the advisor always creates value, since no crowding out of private information occurs and a partially informed decision is strictly better than a completely uninformed one. Note, however, that given the SEC 2003 rule, an institutional investor that does not acquire any information and votes uninformatively, potentially exposes itself to legal risk for violating its fiduciary duty of voting in the best interests of its clients. Given that, it is plausible to assume that even in the absence of a proxy advisor, some private information acquisition would occur. Similarly, the case $c < \underline{c}$ is not empirically plausible because in practice many shareholders voted uninformatively prior to the emergence of proxy advisory firms.

To measure the quality of decision-making, we use the equilibrium expected value of the proposal per-share. The proof of Proposition 1 shows that the expected value of the proposal in the absence of the advisor is given by

$$V_0 = \sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \Lambda, N, k\right) - \frac{1}{2}. \tag{7}$$

3 Voting with the proxy advisory firm

In this section, we introduce the advisor and solve for the equilibria in the game, taking as given fee $f > 0$ set by the advisor (we analyze the fee that maximizes the advisor's profits in the next section). We solve the model by backward induction. First, we find the equilibria at the voting stage. Next, we consider the sale of information stage and solve for the equilibrium information acquisition decisions of the shareholders.

3.1 Voting stage

Following the same argument as in Section 2.1, if a shareholder acquires exactly one signal (private or proxy advisor's), he follows it with probability one. Otherwise, the value of this signal to the shareholder would be zero and he would be better off not paying for it in the first place.

Note also that there is no symmetric equilibrium in which a shareholder acquires both his private signal and the proxy advisor's signal. Indeed, suppose, for example, that such a shareholder votes "for" when the proxy advisor's signal is good and his private signal is bad. The symmetry assumption then implies that when the situation is reversed, i.e., the proxy advisor's signal is bad and his private signal is good, the shareholder votes "against." This, however, implies that the shareholder ignores his private signal and hence would be strictly better off if he did not pay for the private signal. The proof of Proposition 2 presents this argument in more detail.

Given these observations, for information acquisition decisions to be consistent with equilibrium, the equilibrium at the voting stage must take the following form: A shareholder who acquired a private signal votes according to his signal, a shareholder who acquired the advisor's recommendation votes according to the recommendation, and a shareholder who acquired neither signal randomizes between voting "for" and "against" with equal probabilities. The following proposition summarizes the necessary and sufficient conditions for the existence of this equilibrium.

Proposition 2 (voting with the advisor). *In equilibrium, shareholders' strategies at the voting stage must be $w_s(s_i) = s_i$, $w_r(r) = r$, and $w_0 = \frac{1}{2}$. For given q_r, q_s, this equilibrium at the voting stage exists if and only if*

$$\frac{\pi}{1-\pi}\frac{P\left(\frac{1}{2}+\frac{q_r}{2}+q_s(p-\frac{1}{2}),N-1,\frac{N-1}{2}\right)}{P\left(\frac{1}{2}+\frac{q_r}{2}-q_s(p-\frac{1}{2}),N-1,\frac{N-1}{2}\right)}\geq 1. \tag{8}$$

The intuition for condition (8) is as follows. Consider a shareholder who acquired the advisor's recommendation and his incentives to follow it. A rational shareholder understands that his vote only makes a difference if the votes of other shareholders split equally and hence conditions his decision on this event. If $q_r > 0$, a fraction of other shareholders vote based on the advisor's recommendation, and hence the fact that the vote is split implies that a sufficiently high fraction of shareholders who did not acquire the advisor's recommendation vote in the direction opposite this recommendation. In particular, when some shareholders acquire information privately ($q_s > 0$), the vote is more likely to be split when the advisor's recommendation is incorrect ($r \neq \theta$) since a shareholder with private information is more likely to vote against the advisor's recommendation when this recommendation is incorrect than when it is correct. Therefore, as long as $q_r > 0$ and $q_s > 0$, the informational content from the event of being pivotal attenuates the informational effect of the advisor's recommendation and makes it less likely that the shareholder will indeed follow the recommendation. This logic is reflected in (8), which gives the necessary and sufficient condition for a shareholder who got recommendation $r = 1$ to vote for the proposal: the informativeness of the advisor's recommendation is given by $\frac{\Pr[r|\theta=1]}{\Pr[r|\theta=0]}$ and the informativeness of the event that the vote is split is given by $\frac{P(\Pr[v_j=1|r,\theta=1],N-1,\frac{N-1}{2})}{P(\Pr[v_j=1|r,\theta=0],N-1,\frac{N-1}{2})}$, which correspond to the first and second term in (8) when $r = 1$. In particular, condition (8) implies that if some shareholders acquire private information ($q_s > 0$), there exists an upper bound on how many shareholders will acquire the advisor's recommendation ($q_r \leq \bar{q}_r$).

3.2 Information acquisition stage

Having solved for the equilibrium at the voting stage, we calculate the value of information to a shareholder for given q_r and q_s. Using the same arguments as in Section 2.1, we show in the appendix that the values to shareholder i from acquiring a private signal and the recommendation of the advisor are, respectively, given by

$$V_s(q_r,q_s) = (p-\frac{1}{2})\left(\pi\Omega_1(q_r,q_s)+(1-\pi)\Omega_2(q_r,q_s)\right) \tag{9}$$

$$V_r(q_r,q_s) = \frac{1}{2}\left(\pi\Omega_1(q_r,q_s)-(1-\pi)\Omega_2(q_r,q_s)\right), \tag{10}$$

where the expressions $\Omega_1(q_r, q_s) \equiv P\left(\frac{1}{2} + \frac{q_r}{2} + q_s(p - \frac{1}{2}), N-1, \frac{N-1}{2}\right)$ and $\Omega_2(q_r, q_s) \equiv P\left(\frac{1}{2} - \frac{q_r}{2} + q_s(p - \frac{1}{2}), N-1, \frac{N-1}{2}\right)$ denote the probabilities that the votes of other $N-1$ shareholders are split when the advisor's recommendation is correct ($r = \theta$) and when it is incorrect ($r \neq \theta$), respectively. The intuition again follows from the fact that whether a shareholder is informed or not only makes a difference if the votes of other shareholders are split, i.e., if the shareholder's vote is pivotal for the outcome. First, consider (9). Since all other signals are conditionally independent of the shareholder's private signal, the value of the signal to the shareholder equals the probability that the shareholder is pivotal times the value of the signal in this case. The term in the second brackets reflects the probability that the shareholder is pivotal, and $p - \frac{1}{2}$ reflects the value of the signal to the shareholder in this case. Second, consider (10). Now, as long as $q_r > 0$, the acquired signal is no longer conditionally independent of other shareholders' votes because other shareholders acquire the advisor's recommendation as well. When the advisor is correct (incorrect), the value to the shareholder from buying and following the advisor's recommendation conditional on being pivotal is $\frac{1}{2}$ $(-\frac{1}{2})$ because the shareholder makes the correct (incorrect) decision instead of randomizing between them with probability $\frac{1}{2}$.

A shareholder is better off acquiring the private signal than staying uninformed if and only if $V_s(q_r, q_s) \geq c$, and is better off acquiring the advisor's recommendation than staying uninformed if and only if $V_r(q_r, q_s)$ exceeds fee f that the advisor charges. Given (9) and (10), we can determine the equilibrium information acquisition strategies. If $q_r > 0$, i.e., some shareholders acquire the advisor's recommendation, the following two cases are possible:

- **Case 1: Incomplete crowding out of private information acquisition.** Shareholders randomize between acquiring the advisor's recommendation, the private signal, and staying uninformed: $q_r > 0$, $q_s > 0$, and $q_s + q_r \leq 1$.[13] In this case, q_r and q_s are found from

$$V_s(q_r, q_s) - c = V_r(q_r, q_s) - f \geq 0, \tag{11}$$

  with equality if $q_s + q_r < 1$.

- **Case 2: Complete crowding out of private information acquisition.** No shareholder acquires the private signal: $q_s = 0$. In this case, shareholders randomize between

[13] More specifically, if $q_s + q_r < 1$, shareholders randomize between acquiring the advisor's recommendation, acquiring the private signal, and staying uninformed, and if $q_s + q_r = 1$, all shareholders become informed and randomize between acquiring the advisor's recommendation and the private signal.

acquiring the advisor's recommendation and staying uninformed. Probability q_r is given by $V_r(q_r, 0) = f$, which implies

$$q_r = \sqrt{1 - 4\left(\frac{f}{\pi - \frac{1}{2}} \frac{1}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}. \tag{12}$$

For this to be an equilibrium, it must be that $V_s(q_r, 0) \leq c$.

The next lemma summarizes the set of symmetric equilibria for all values of f.

Lemma 1. *Let* $\underline{f} \equiv \frac{c}{2p-1} - C_{N-1}^{\frac{N-1}{2}} 2^{1-N}(1-\pi)$ *and* $\bar{f} \equiv \frac{2\pi-1}{2p-1}c$. *For a given fee* $f > 0$, *the set of symmetric Bayes-Nash equilibria is as follows:*

1. If $f \geq \bar{f}$, *there is a unique equilibrium, which is identical to that in the benchmark model. In particular, no shareholder acquires the recommendation from the advisor.*

2. If $f \in [\underline{f}, \bar{f})$, *there exist three equilibria: (a) equilibrium with incomplete crowding out of private information acquisition and* $0 < q_r \leq (2p-1) q_s$; *(b) equilibrium with incomplete crowding out of private information acquisition and* $q_r \geq (2p-1) q_s > 0$; *and (c) equilibrium with complete crowding out of private information acquisition:* $q_s = 0, q_r \in (0, 1)$. *Equilibria (a) and (b) coincide when* $f = \underline{f}$. *These equilibria can be ranked in their shareholder value (expected value of the proposal minus information acquisition costs), with equilibrium (a) having the highest and equilibrium (c) having the lowest shareholder value.*

3. If $f < \underline{f}$, *the unique equilibrium has complete crowding out of private information acquisition:* $q_s = 0, q_r \in (0, 1)$.

The structure of the equilibrium is intuitive. If the advisor's fee is very high, $\frac{f}{2\pi-1} \geq \frac{c}{2p-1}$, no shareholder finds it optimal to acquire its recommendation. Intuitively, this condition implies that the cost f of the advisor's signal relative to its precision π is lower than the cost c of the private signal relative to its precision p. If the advisor's fee is very low, $f < \underline{f}$, no shareholder finds it optimal to acquire private information, and all shareholders randomize between remaining uninformed and buying the advisor's signal. Finally, in the intermediate range of f, there exist equilibria in which both types of signals are acquired in equilibrium. In this region, there are multiple equilibria for the following reason. Recall that given the same probability of being pivotal, the private value from buying the advisor's recommendation is the highest when either no shareholder acquires the advisor's signal or no shareholder acquires private information. Therefore, shareholders' decisions to acquire the advisor's recommendation instead of private signals are strategic substitutes when few shareholders rely on the advisor, but become strategic complements when many shareholders rely on the advisor. As a consequence, multiple equilibria exist when the advisor's fee is in the intermediate range.

In what follows, we assume that when the advisor's fee is in the intermediate range, $f \in [\underline{f}, \bar{f})$, shareholders coordinate on the equilibrium in which shareholder value is maximized. Since shareholders are identical, this selection is identical to the Pareto-dominance criterion, according to which an equilibrium is not selected if there exists another equilibrium with higher payoffs for all players in the subgame.

Assumption 2 (equilibrium selection). *When multiple equilibria exist at the information acquisition stage, shareholders coordinate on the equilibrium that maximizes shareholder value, defined as the expected value of the proposal minus expected information acquisition costs of shareholders.*

Assumption 2 makes the pricing problem of the seller, studied in the next section, well-defined. Importantly, however, as we discuss below and show in Proposition 4, it is not necessary for our main results about the advisor's effect on the efficiency of decision-making.

Assumption 2 and Lemma 1 imply the following equilibrium in the information acquisition subgame:

Proposition 3 (equilibrium information acquisition). *For a given fee f, the equilibrium at the information acquisition stage is as follows:*

1. *If $f \geq \bar{f}$, then $q_r = 0$ and $q_s = q_0^* \in (0,1)$, given by (6).*

2. *If $f \in [\underline{f}, \bar{f})$, then $q_r \in (0, (2p-1)\, q_s]$ and $q_s \in (0, 1 - q_r]$, which satisfy (11), with strict equality if $q_s + q_r < 1$, and are given by (26) in the Appendix.*

3. *If $f < \underline{f}$, then $q_s = 0$ and $q_r \in (0,1)$, given by (12).*

Figure 2 illustrates Proposition 3. In this example, there are 35 shareholders, the private information acquisition cost is 1.5% of the potential value of the proposal per shareholder, and the precisions of the private signal and the advisor's recommendation are $p = 0.65$ and $\pi = 0.75$, respectively. When the advisor's fee exceeds $\bar{f} = 2.5\%$, the precision-to-price ratio of the advisor's signal is below that of the private signal. In this case, no shareholder acquires information from the advisor, and the equilibrium is identical to the benchmark case. In particular, a shareholder acquires a private signal with probability 44.5% and remains uninformed with probability 55.5%. When the advisor's fee is between $\underline{f} \approx 1.6\%$ and $\bar{f} = 2.5\%$, incomplete crowding out of private information acquisition occurs in equilibrium. In this range, as fee f decreases, the probability that a shareholder acquires the advisor's recommendation (private signal) increases (decreases), and the probability that a shareholder remains uninformed increases. Finally, when the fee charged by the advisor is below $\underline{f} \approx 1.6\%$, private information becomes relatively costly, so the advisor completely crowds out private information acquisition. As the fee declines even more, the probability with which a shareholder becomes informed by buying the advisor's recommendation (stays uninformed) increases (decreases).

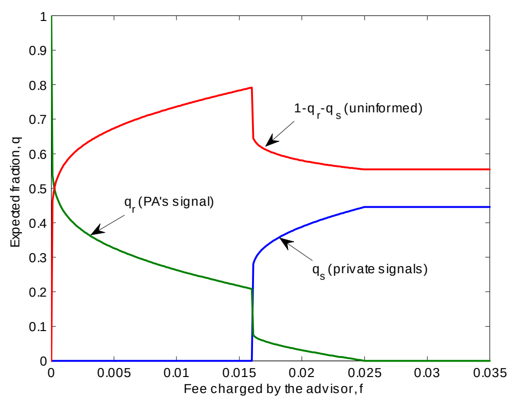

Figure 2. Equilibrium information acquisition. The figure plots the equilibrium information information acquisition as a function of the fee f charged by the advisor. The blue line depicts the equilibrium probability q_s that a shareholder acquires his private signal. The green line depicts the equilibrium probability q_r that a shareholder acquires the recommendation from the advisor. The red line depicts the equilibrium probability that a shareholder remains uninformed. The parameters are $N = 35$, $p = 0.65$, $\pi = 0.75$, and $c = 0.015$.

3.3 Quality of decision-making

Given the equilibrium at the information acquisition and voting stages, we can compute the per-share expected value of the proposal, which measures the quality of decision-making with the advisor. Comparing it with value (7) in the benchmark case allows us to examine whether the presence of the advisor increases firm value for a given fee f. The following proposition is the main result of the paper:

Proposition 4 (quality of decision-making for a given fee). *Fix fee f.*

1. *In any equilibrium with incomplete crowding out of private information acquisition, firm value is strictly lower than in the benchmark case.*

2. *Consider equilibrium with complete crowding out of private information acquisition. There exists threshold $\pi^*(f) \in [\frac{1}{2} + \frac{f}{c}(p - \frac{1}{2}), 1]$, such that firm value is lower than in the benchmark case if and only if $\pi \leq \pi^*(f)$.*

Proposition 4 shows that the presence of the advisor harms the quality of decision-making unless there is complete crowding out of private information acquisition and the advisor's signal is sufficiently precise. Intuitively, this happens because the information acquisition decision that is privately optimal from a shareholder's perspective is not socially optimal: a shareholder does not internalize the externality that his decision to follow the advisor's recommendation has on other shareholders. As a result, there is inefficient crowding out of private information acquisition, leading to suboptimal voting decisions.

To see the intuition in the simplest way, consider the second part of the proposition, i.e., the case of complete crowding out of private information production, and suppose that $f = c$. Consider a shareholder's decision whether to acquire his own private signal at cost c or to buy the advisor's signal at cost f. Being rational, the shareholder conditions his decision on the event that his vote makes a difference, i.e., the votes of other shareholders are split. Because no other shareholder acquires private information, the vote can only be split if there are sufficiently many uninformed shareholders who vote against the advisor's recommendation. However, because these shareholders' votes are uninformed, this information does not add anything to the shareholder's prior beliefs about the informativeness of the advisor's recommendation. Hence, conditional on being pivotal, the value from voting according to the advisor's recommendation is $\pi - \frac{1}{2}$, and the value from voting according to a private signal is $p - \frac{1}{2}$. Given that the two signals are equally costly, the shareholder finds it privately optimal to acquire the advisor's signal if it is more precise, $\pi > p$, and acquire his private signal if $\pi < p$. In particular, the shareholder's privately optimal choice does not take into account how many other shareholders acquire the advisor's recommendation: as long as $\pi > p$, it is optimal for him to buy the advisor's signal instead of the private signal even if many other shareholders follow the advisor as well.

This, however, is socially inefficient. Indeed, if many shareholders are following the advisor's recommendation, they all vote in the same way, and their mistakes are perfectly correlated. In contrast, when shareholders are following their private signals, their mistakes are independent (or, in a more general setting, imperfectly correlated) conditional on the state, and hence the voting outcome is more likely to be efficient. Formally, Proposition 4 shows that the expected value of the proposal is higher in the equilibrium with complete crowding out than in the equilibrium without the advisor if and only if the advisor's signal is sufficiently precise, $\pi > \pi^*(f)$. The intuition for the case of incomplete crowding out is similar, although a bit more involved.

Importantly, the result that the presence of the advisor can be detrimental for firm value crucially depends on the coordination problem due to collective decision-making by shareholders. If the firm had only one shareholder or if shareholders could coordinate their information acquisition and voting decisions, the presence of an additional valuable signal from the advisor would always be beneficial.

4 Pricing of information by the proxy advisor

In this section, we study strategic fee setting by the monopolistic advisor. The advisor maximizes its profits, taking into account how its fee affects shareholders' information acquisition decisions. Proposition 3 implies that the demand function for the advisor's recommendation is given by

$$q_r(f) = \begin{cases} q_r^H(f), & \text{if } f < \underline{f}, \\ q_r^L(f), & \text{if } f \in [\underline{f}, \bar{f}), \\ 0, & \text{if } f \geq \bar{f}, \end{cases} \tag{13}$$

where $q_r^H(f)$ corresponds to complete crowding out of private information and is given by (12), and $q_r^L(f) < q_r^H(f)$ corresponds to incomplete crowding out of private information and is given by (26) in the Appendix. An example of this demand function is shown in Figure 2. The optimal fee chosen by the advisor, denoted f^*, maximizes its expected revenues $fq_r(f)$.

Consider the unconstrained problem of the advisor, $f = \arg\max fq_r^H(f)$, i.e, the problem where the advisor faces no competition from the private information acquisition technology. The proof of Proposition 5 shows that the function $fq_r^H(f)$ is inverse U-shaped in f and has a maximum at

$$f_m \equiv (\pi - \frac{1}{2})P(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N-1, \frac{N-1}{2}), \tag{14}$$

which corresponds to $q_r = \frac{1}{\sqrt{N}}$.

It follows that depending on the parameters, one of the following three cases is possible. If $f_m < \underline{f}$, which happens when the precision of the advisor's signal is sufficiently high and the private information acquisition technology is sufficiently costly, then the advisor sets $f^* = f_m$. If $f_m \geq \underline{f}$, then one of the two scenarios is possible. First, the advisor could select the maximum possible fee given which there is complete crowding out of private information acquisition. This strategy is akin to "limit pricing" in industrial organization, where the incumbent sets its price just low enough to make it unprofitable for a potential entrant to enter the market. Second, the advisor could select fee $f^* > \underline{f}$ that maximizes its revenues conditional on incomplete crowding out of private information acquisition.

Denote $V^*(\pi)$ the expected value of the proposal given the equilibrium fee f^* chosen by the advisor. Under what conditions is $V^*(\pi)$ higher than in the benchmark model without the advisor? Lemma 1 and Proposition 4 imply that it can happen only if the advisor chooses fee f^* that maximizes its unconstrained problem, i.e., if $f^* = f_m < \underline{f}$ (see the proof of Proposition 5 for details). In other words, firm value can only be higher than in the benchmark case if the advisor sets fee $f^* = f_m$, and each shareholder acquires the advisor's signal with probability $\frac{1}{\sqrt{N}}$ and remains uninformed otherwise. The expected value of the proposal in this case is given by

$$V^*(\pi) = (\pi - \frac{1}{2})[2\sum_{k=\frac{N+1}{2}}^{N} P(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k) - 1]. \tag{15}$$

To compare it with firm value in the benchmark case, which is given by V_0 in (7), define $\pi^* \equiv \sum_{k=\frac{N+1}{2}}^{N} P(p_0, N, k)$, where $p_0 \equiv pq_0^* + \frac{1-q_0^*}{2}$ and q_0^* is the benchmark equilibrium probability of a shareholder acquiring private information, given by (6). Intuitively, π^* is the equilibrium probability of making a correct decision in the benchmark model without the advisor. Then $V_0 = \pi^* - \frac{1}{2}$, and hence condition $V^*(\pi) > V_0$ holds if and only if

$$\pi > \tilde{\pi} \equiv \frac{1}{2} + \frac{\pi^* - \frac{1}{2}}{2\sum_{k=\frac{N+1}{2}}^{N} P(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k) - 1}. \tag{16}$$

Interestingly, since the denominator in the second term of (16) is below one, $\tilde{\pi}$ exceeds one if π^* is sufficiently high, that is, if private signals are relatively cheap and a sufficient fraction of shareholders acquires information in the benchmark case. In this case, the advisor always harms firm value, even if $\pi = 1$, i.e., its information is perfectly precise. Intuitively, even if its recommendation is extremely precise, the advisor never finds it optimal to sell it to all shareholders: its profits are higher if it sells the recommendation to fewer shareholders but charges a higher fee. As a consequence, many shareholders remain uninformed and hence the advisor's information does not get perfectly incorporated in the vote. If the efficiency of decision-making without the advisor is sufficiently high, this effect implies that the presence of the advisor harms firm value even if the advisor is perfectly informed. These results are summarized in the following proposition.

Proposition 5 (equilibrium quality of decision-making). *Firm value in the presence of the advisor is strictly lower than in the benchmark case if and only if the precision of the advisor's signal π is below $\tilde{\pi}$ given by (16). In particular, if $(2p-1)\,q_0^* > \frac{1}{\sqrt{N}}$, then firm value is strictly lower than in the benchmark case for any precision $\pi \in (\frac{1}{2}, 1]$ of the advisor's signal.*

Figure 3 illustrates how the equilibrium fee charged by the advisor and the expected firm value relative to the benchmark case depend on the precision of the advisor's recommendation. Figures 3a-3c use the same parameters as Figure 2: there are 35 shareholders, the private information acquisition cost is 1.5% of the potential value of the proposal per shareholder, and the precision of the private signal is $p = 0.65$. When the advisor's information is sufficiently precise, $\pi > 0.84$, it can set the fee in a way as if it faced no competition from the private information acquisition technology: $f^* = f_m$, the unconstrained optimal fee. When the advisor's information is less precise, $\pi < 0.84$, shareholders would acquire private information, had the advisor set the fee at f_m. To prevent this, the advisor engages in limit pricing by setting the fee at the highest possible level that allows it to crowd out private information acquisition. As a result of this pricing strategy, shareholders do not acquire private information for any $\pi > 0.64$. Finally, when the precision of the advisor's recommendation falls below 0.64, both types of signals are acquired in equilibrium. Figure 3c illustrates the first statement of Proposition 5 and shows that the expected value of the proposal is higher than in the benchmark case only if there is complete crowding out of private information acquisition and the advisor's signal is sufficiently precise, $\pi > 0.92$. The graph of social welfare, defined as the expected value of the proposal minus shareholders' costs of private information acquisition, looks very similar and is omitted for brevity. In particular, under the above parameters, the presence of the advisor hurts social welfare unless its information is sufficiently precise.

Finally, Figure 3d illustrates the second statement of Proposition 5 and shows that if the shareholders' private signals are sufficiently cheap ($c = 0.75\%$ in this example), the presence of the advisor hurts firm value even if its information is perfectly precise.

Overall, Proposition 5 shows that even taking into account the equilibrium fee set by the advisor, the quality of corporate decisions is reduced if the advisor's signal is not precise enough. In fact, as the results of the next section demonstrate, strategic fee setting by the advisor exacerbates its negative influence when its recommendations are not too precise and decreases its potential positive influence when its recommendations are sufficiently precise.

5 Analysis of regulation

In this section, we analyze two types of proposed regulations in the context of our model – those aimed at reducing proxy advisors' market power and those aimed at increasing transparency.

5.1 Restricting the advisor's market power

It is frequently argued that proxy advisory firms, especially ISS, have too much market power. Indeed, the proxy advisory industry is dominated by two players, ISS and Glass Lewis, who together control 97% of the market in terms of their clients' equity assets, with ISS controlling 61% of the market. As a result, proposals to restrict proxy advisors' market power have been widely discussed (e.g., GAO, 2007; Edelman, 2013). For example, according

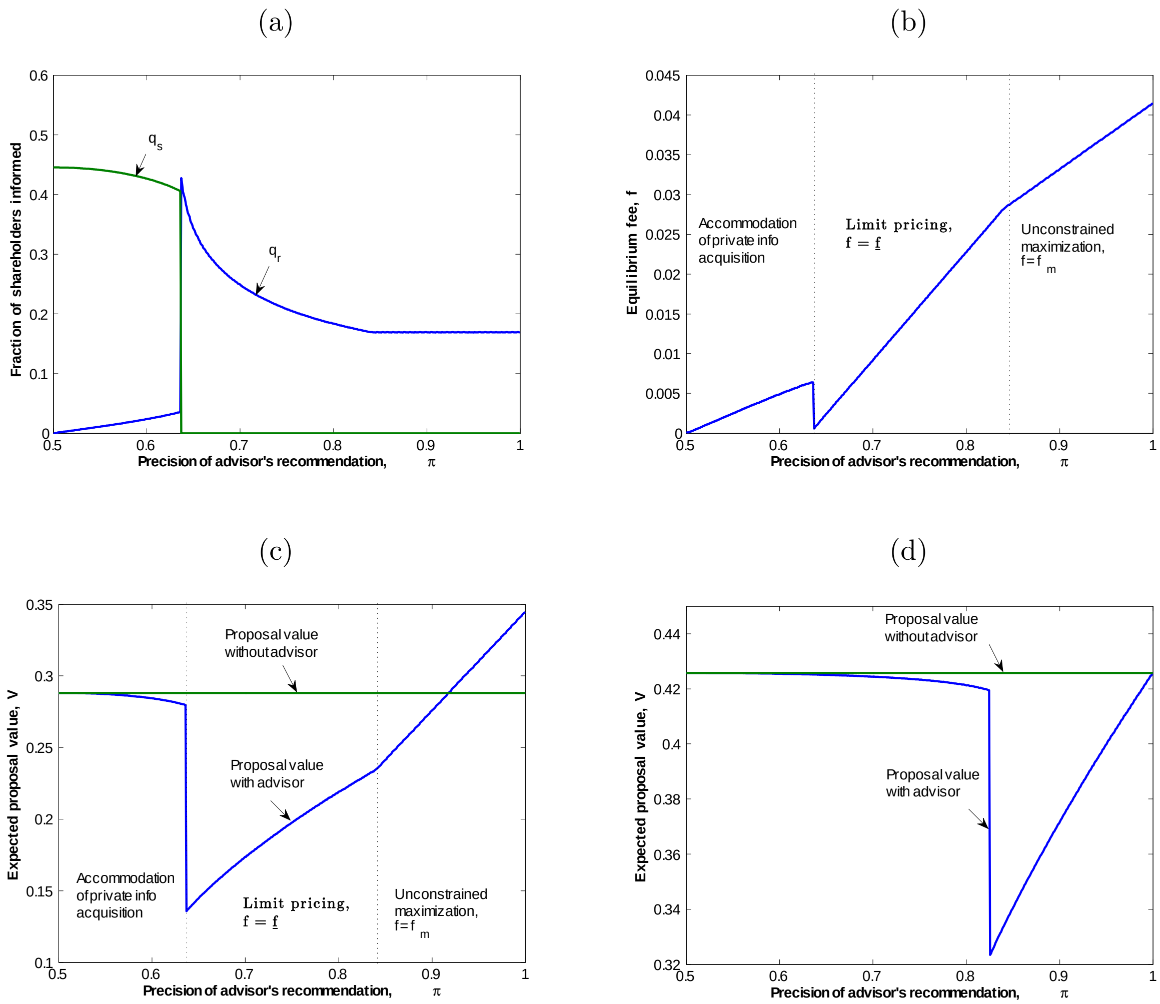

Figure 3. Equilibrium fee, information acquisition decisions, and quality of decision-making for different levels of precision of the advisor's signal. Figure (a) plots the equilibrium probability of a shareholder acquiring the advisor's recommendation (q_r) and a private signal (q_s) as functions of the precision of the advisor's signal π. Figure (b) plots the equilibrium fee set by the advisor as a function of the precision of its recommendation. Figure (c) plots the equilibrium expected value of the proposal and its value in the benchmark case. As one can see, the presence of the advisor harms quality of decision-making unless the precision of its recommendation is precise enough. Figure (d) plots the same figure but when the cost of private information acquisition c is half the baseline amount. The parameters are $N = 35$, $p = 0.65$, $c = 0.015$ (except figure (d)), and $c = 0.0075$ in figure (d).

to the Government Accountability Office report (GAO, 2007), institutional investors believe that reducing ISS's market power could help negotiate better prices with ISS and overall reduce the costs of proxy voting advice.

We can use our model to study the costs and benefits of these proposals within our framework. In particular, consider the effect of a marginal reduction in the fee charged by the advisor from the equilibrium f^* to a lower value. As the next proposition demonstrates, whether such a reduction in market power is beneficial, crucially depends on the equilibrium information acquisition decisions by the shareholders, an in particular, on how much private information they acquire. To see this, suppose first that given the equilibrium fee f^*, shareholders do not acquire any private information. Conditional on having complete crowding out of private information, it is optimal (for the quality of decision-making) that more shareholders follow the advisor, since following the advisor dominates uninformed voting. Therefore, if complete crowding out of private information acquisition occurs in equilibrium, a marginal reduction of the advisor's fee makes shareholders more informed and increases firm value. In contrast, if some shareholders acquire private information in equilibrium, a reduction in the advisor's fee has the negative effect of crowding out some of this private information acquisition. By the same logic as in Proposition 4, this additional crowding out of private information acquisition is inefficient and lowers the quality of decision-making. The following result formalizes these arguments:

Proposition 6 (restricting market power). *A marginal reduction in the advisor's fee increases firm value if equilibrium features complete crowding out of private information acquisition, but decreases firm value if equilibrium features incomplete crowding out of private information acquisition.*

Proposition 6 implies that restricting the advisor's market power will lead to more efficient voting outcomes only if the advisor's information is sufficiently precise, so that there is no private information acquisition and too little acquisition of the advisor's recommendations by the shareholders. In contrast, if the advisor's information is not too precise, decreasing its market power decreases the quality of decision-making because it encourages even more overreliance on its recommendations and exacerbates the inefficient crowding out of private information production.

The next proposition illustrates this intuition by answering a more general question: If one could choose the fee that the advisor charges for its recommendations, what fee would maximize firm value? Consistent with the arguments above, if the advisor's information is not too precise, it would be optimal to make its recommendations prohibitively expensive to deter shareholders from buying them all together (Lemma 1 implies that any fee $f \geq \bar{f}$ would achieve this). In contrast, if the advisor's information is sufficiently precise, it would be optimal to set the fee at the lowest possible level to encourage as many shareholders as possible to buy the advisor's recommendations.[14]

Proposition 7 (fee that maximizes firm value). *Let f_{opt} be the fee that maximizes the expected value of the proposal. Then $f_{opt} \geq \bar{f}$ if $\pi \leq \pi^*$, and f_{opt} is arbitrarily close to zero if $\pi > \pi^*$, where $\pi^* \equiv \sum_{k=\frac{N+1}{2}}^{N} P(pq_0^* + \frac{1-q_0^*}{2}, N, k)$ and q_0^* is given by (6).*

5.2 Disclosing the quality of the advisor's recommendations

Another frequently discussed policy is to increase the transparency of proxy advisors' methodologies and procedures to make it easier for investors to evaluate the quality of their recommendations. This includes both disclosure of potential conflicts of interest (which might arise if the proxy advisor provides consulting services to corporations) and disclosure of assumptions and sources of information underlying their recommendations. For example, the 2010 SEC concept release on the U.S. proxy system puts forward a proposal that would require proxy advisors to "provide increased disclosure regarding the extent of research involved with a particular recommendation and the extent and/or effectiveness of its controls and procedures in ensuring the accuracy of issuer data." With respect to conflicts of interest, the 2014 SEC Staff Legal Bulletin No. 20 requires that proxy advisors disclose potential conflicts of interest to their existing clients, but many market participants push for further regulation, which would require conflicts of interests to be disclosed to the broader public.

In this section, we examine the potential effects of such proposals in the context of our model. Specifically, consider the following modification of our baseline setting. The actual precision of the advisor's signal can be high or low, $\pi \in \{\pi_l, \pi_h\}$, $\pi_l < \pi_h$, with probabilities μ_l and μ_h, $\mu_h + \mu_l = 1$. For example, $\pi = \pi_l$ can capture the precision of the advisor's

[14] We obtain Proposition 7 under the simplifying assumption that the advisor is endowed with information, i.e., that we do not need to satisfy the advisor's participation constraint. If the advisor has a cost $c_A > 0$ of producing its recommendation, a similar result holds, but with a different cutoff $\tilde{\pi}^*$ and the optimal fee f_{opt} that just compensates the advisor for producing its recommendation when $\pi \leq \tilde{\pi}^*$.

signal for companies where it has conflicts of interest, while $\pi = \pi_h$ can capture the higher precision for companies where it has no conflicts of interest. Let $\bar{\pi} \equiv \mu_l \pi_l + \mu_h \pi_h$ denote the expected precision of the signal.

We compare the quality of decision-making in two regimes – when the precision of the advisor's signal is publicly disclosed and when it remains unknown to the shareholders. If the precision of the advisor's signal is disclosed, the timing of the game is as follows. First, precision $\pi \in \{\pi_l, \pi_h\}$ is realized and learned by all parties. Then, the advisor decides on the fee it charges for its recommendation. After that, shareholders non-cooperatively decide what signals to acquire and how to vote. If the precision of the advisor's signal is not disclosed, the timing of the game is identical to that in the previous sections: The advisor sets the fee it charges, shareholders decide what signal to acquire, not knowing whether $\pi = \pi_l$ or $\pi = \pi_h$, and then decide how to vote. The proof of Proposition 8 shows that the equilibrium in this game coincides with the equilibrium of the basic model for $\pi = \bar{\pi}$.

We make a simplifying assumption that uncertainty about the precision of the advisor's signal is rather high:

Assumption 3 (high precision uncertainty). *$\pi_l = \frac{1}{2}$ and π_h is such that complete crowding out of private information acquisition occurs in equilibrium of the basic model with $\pi = \pi_h$.*

Assumption 3 implies that if the quality of the advisor's information is low, its signal is completely uninformative. Clearly, if shareholders know that the advisor's signal is pure noise, no shareholder buys it, and the equilibrium is identical to the benchmark model without the advisor. In contrast, if the quality of the advisor's information is high and shareholders know about it, no shareholder acquires private information.

The next proposition gives sufficient conditions under which disclosure improves the quality of decision-making:

Proposition 8 (disclosure of precision). *Firm value is strictly higher when the precision of the advisor's signal is disclosed if at least one of the following conditions is satisfied:*

1. *$V^*(\pi_h) > V_0$, i.e., firm value is higher with the advisor than without when $\pi = \pi_h$;*
2. *Complete crowding out of private information acquisition occurs when $\pi = \bar{\pi}$.*

The intuition behind this result is as follows. Disclosing the precision of the advisor's recommendations allows shareholders to tailor their information acquisition decisions to the quality of the recommendations: shareholders do not acquire the advisor's recommendations if $\pi = \frac{1}{2}$ and do not acquire private information if $\pi = \pi_h$. Under the first condition in Proposition 8, such tailored information acquisition decisions are rather efficient: they ensure that the advisor's recommendations do not affect the vote when they are uninformative, and that they have a relatively large effect on the vote when they are sufficiently informative ($V^*(\pi_h) > V_0$). Hence, disclosure leads to more informed voting decisions than if shareholders made their decisions based on the average precision $\bar{\pi}$ and sometimes relied on the advisor's recommendations when they are completely uninformative. A similar argument applies under the second condition in Proposition 8: without disclosure, shareholders do not acquire private information and completely rely on the advisor's recommendations, even though they are sometimes uninformative. In contrast, with disclosure, shareholders perform independent research when the advisor's recommendations are uninformative, leading to more informed voting decisions.

Interestingly, however, disclosing the precision of the advisor's recommendations does not always improve the quality of decision-making: Disclosure may encourage even stronger crowding out of private information acquisition and decrease firm value. To see this, consider the numerical example of Figure 3 and suppose that $\pi_l = \frac{1}{2}$, $\pi_h = 0.7$, and $\mu_l = \mu_h = \frac{1}{2}$, so that $\bar{\pi} = 0.6$. Without disclosure, expected firm value is given by $V^*(0.6)$, which, as Figure 3c demonstrates, is very close to value V_0 in the benchmark case without the advisor. This is because the expected precision of the advisor's signal is sufficiently low, so that there is relatively little crowding out of private information acquisition in equilibrium. In contrast, with disclosure, expected firm value is the average of V_0 and $V^*(0.7)$, and this average is lower than $V^*(0.6)$. Thus, in this example, disclosure makes voting decisions less informed and decreases firm value. The reason is that when $\pi = \pi_h$, the advisor's recommendations are not precise enough to improve decision-making but are sufficiently precise to completely crowd out private information acquisition. This inefficient crowding out of private information when $\pi = \pi_h$ is detrimental for firm value, and even the more efficient decision-making when $\pi = \pi_l$ is not sufficient to counteract its negative effect.

6 Conclusion

Proxy advisors are playing an increasingly important role in corporate governance by providing institutional investors with governance research and recommendations on how to vote their shares: instead of conducting costly independent research, investors can buy information from proxy advisors for a fee. The goal of this paper is to examine the effect of proxy advisors on the quality of corporate decisions and to evaluate the existing policy proposals on regulating the proxy advisory industry. We develop a model of strategic shareholder voting, in which a monopolistic advisor (proxy advisory firm) offers to sell its information (vote recommendations) to voters (shareholders) for a fee, and voters non-cooperatively decide whether to engage in private information production and/or buy the advisor's recommendation, and how to cast their votes.

We show that even if the proxy advisor's recommendations are completely unbiased and more informative than the research each shareholder could do on his own, the advisor's presence can make shareholder votes less informed and thereby decrease the quality of corporate decisions. This is because there is a wedge between the private and social value of information in voting, resulting in inefficient crowding out of private information acquisition. Intuitively, when many shareholders follow the proxy advisor, they make perfectly correlated voting mistakes, and hence the vote would be more efficient if shareholders acquired and followed their private signals instead. However, each individual shareholder deciding between acquiring his own information and buying the proxy advisor's recommendation fails to internalize the externality he imposes on other shareholders, and hence his privately optimal information acquisition decision is different from the socially optimal one. As a result, private information production is inefficiently crowded out, leading to less informed voting decisions and lower firm value. Overall, in our setting, the presence of the proxy advisor positively affects the quality of corporate decisions if and only if its information is sufficiently precise. Moreover, if the quality of decision-making without the advisor is sufficiently high, then the advisor's presence decreases firm value even if its information is perfectly precise.

We also examine the effects of several proposals that have been put forward to regulate the proxy advisory industry. For example, in our setting, reducing the advisor's market power and decreasing the price of its recommendations is only beneficial if the advisor's information is sufficiently precise, but has a negative effect if it is not precise enough. We also show that improved disclosure about the quality of the advisor's recommendations can have both a positive and negative effect on firm value, depending on the quality of their information relative to the private information of the shareholders. More generally, our analysis implies that the costs and benefits of regulatory proposals crucially depend on how informative proxy advisors' recommendations are and how the regulation will affect private information production by investors.

References

[1] Admati, Anat R., and Paul Pfleiderer (1986). A monopolistic market for information, *Journal of Economic Theory* 39, 400-438.

[2] Admati, Anat R., and Paul Pfleiderer (1990). Direct and indirect sale of information, *Econometrica* 58, 901-928.

[3] Alexander, Cindy R., Mark A. Chen, Duane J. Seppi, and Chester S. Spatt (2010). Interim news and the role of proxy voting advice, *Review of Financial Studies* 23, 4419-4454.

[4] Alonso, Ricardo, and Odilon Câmara (2016). Persuading voters. *American Economic Review*, forthcoming.

[5] Austen-Smith, David, and Jeffrey S. Banks (1996). Information aggregation, rationality, and the Condorcet Jury Theorem, *American Political Science Review* 90, 34-45.

[6] Bergemann, Dirk, Alessandro Bonatti, and Alex Smolin (2015). Designing and pricing information, Working Paper.

[7] Bond, Philip, and Hulya Eraslan (2010). Strategic voting over strategic proposals, *Review of Economic Studies* 77, 459-490.

[8] Bond, Philip, and Itay Goldstein (2015). Government intervention and information aggregation by prices, *Journal of Finance* 70, 2777-2812 .

[9] Boot, Arnoud W. A., and Anjan V. Thakor (2001). The many faces of information disclosure, *Review of Financial Studies* 14, 1021-1057.

[10] Brav, Alon, and Richmond D. Mathews (2011). Empty voting and the efficiency of corporate governance, *Journal of Financial Economics* 99, 289-307.

[11] Cespa, Giovanni (2008). Information sales and insider trading with long-lived information, *Journal of Finance* 63, 639-672.

[12] Chakraborty, Archishman, and Rick Harbaugh (2010). Persuasion by cheap talk. *American Economic Review* 100, 2361-2482.

[13] Cuñat, Vicente, Mireia Gine, and Maria Guadalupe (2012). The vote is cast: the effect of corporate governance on shareholder value, *Journal of Finance* 67, 1943-1977.

[14] Diamond, Douglas W. (1985). Optimal release of information by firms, *Journal of Finance* 40, 1071-1094.

[15] Edelman, Sagiv (2013). Proxy advisory firms: A guide for regulatory reform, *Emory Law Journal* 62, 1369-1409.

[16] Ertimur, Yonca, Fabrizio Ferri, and David Oesch (2013). Shareholder votes and proxy advisors: evidence from say on pay, *Journal of Accounting Research* 51, 951-996.

[17] Ertimur, Yonca, Fabrizio Ferri, and Stephen Stubben (2010). Board of directors' responsiveness to shareholders: Evidence from shareholder proposals, *Journal of Corporate Finance* 16, 53-72.

[18] Feddersen, Timothy, and Wolfgang Pesendorfer (1998). Convicting the innocent: the inferiority of unanimous jury verdicts under strategic voting. *American Political Science Review* 92, 23-35.

[19] Fishman, Michael J., and Kathleen M. Hagerty (1995). The incentive to sell financial market information, *Journal of Financial Intermediation* 4, 95-115.

[20] Garcia, Diego, and Francesco Sangiorgi (2011). Information sales and strategic trading, *Review of Financial Studies* 24, 3069-3104.

[21] Gerardi, Dino, and Leeat Yariv (2008). Information acquisition in committees, *Games and Economic Behavior* 62, 436-459.

[22] Gershkov, Alex, and Balázs Szentes (2009). Optimal voting schemes with costly information acquisition, *Journal of Economic Theory* 144, 36-68.

[23] Government Accountability Office (GAO) report "Corporate shareholder meetings: Issues relating to firms that advise institutional investors on proxy voting," June 2007.

[24] Hellwig, Christian, and Laura Veldkamp (2009). Knowing what others know: Coordination motives in information acquisition, *Review of Economic Studies* 76, 223-251.

[25] Iliev, Peter, and Michelle Lowry (2015). Are mutual funds active voters? *Review of Financial Studies* 28, 446-285.

[26] Jackson, Matthew O., and Xu Tan (2013). Deliberation, disclosure of information, and voting. *Journal of Economic Theory* 148, 2-30.

[27] Khanna, Naveen, and Mark Schroder (2015). "Rewarding disagreement" for optimal collective decisions: The role of skill and luck, Working Paper.

[28] Larcker, David F., Allan L. McCall, and Gaizka Ormazabal (2015). Outsourcing shareholder voting to proxy advisory firms, *Journal of Law and Economics* 58, 173-204.

[29] Levit, Doron, and Nadya Malenko (2011). Nonbinding voting for shareholder proposals, *Journal of Finance* 66, 1579-1614.

[30] Li, Tao (2014). Outsourcing corporate governance: conflicts of interest within the proxy advisory industry, Working Paper.

[31] Malenko, Nadya, and Yao Shen (2016). The role of proxy advisory firms: Evidence from a regression-discontinuity design, Working Paper.

[32] Martinelli, César (2006). Would rational voters acquire costly information? Journal of Economic Theory 129, 225-251.

[33] Maug, Ernst (1999). How effective is proxy voting? Information aggregation and conflict resolution in corporate voting contests, Working Paper.

[34] Maug, Ernst, and Bilge Yilmaz (2002). Two-class voting: A mechanism for conflict resolution, *American Economic Review* 92, 1448-1471.

[35] Maug, Ernst, and Kristian Rydqvist (2009). Do shareholders vote strategically? Voting behavior, proposal screening, and majority rules, *Review of Finance* 13, 47-79.

[36] Morris, Stephen, and Hyun Song Shin (2002). Social value of public information, *American Economic Review* 92, 1521-1534.

[37] Persico, Nicola (2004). Committee design with endogenous information, *Review of Economic Studies* 71, 165–191.

[38] Schnakenberg, Keith E. (2015). Expert advice to a voting body. *Journal of Economic Theory* 160, 102-113.

Appendix: Proofs

Proof of Proposition 1.

Fix probability q with which each shareholder i acquires a private signal s_i. We start by proving that for any q, the equilibrium $w_s(0) = 0$, $w_s(1) = 1$, and $w_0 = \frac{1}{2}$ exists (as argued before, this is the only possible equilibrium at the voting stage because otherwise information would have zero value and acquiring it would be suboptimal). Consider the decision of shareholder i with signal s_i when other informed shareholders (i.e., shareholders that acquired private signals) vote according to strategy $w_s(s_j)$, and uninformed shareholders (i.e., shareholders that did not acquire private signals) vote according to strategy $w_0 = \frac{1}{2}$. Given q, the probability that each shareholder votes "for" in state $\theta \in \{0, 1\}$ equals

$$\begin{gathered}\Pr[v_j = 1|\theta = 1] = q(w_s(1)p + w_s(0)(1-p)) + (1-q)\tfrac{1}{2} = qp + (1-q)\tfrac{1}{2}, \\ \Pr[v_j = 1|\theta = 0] = q(w_s(1)(1-p) + w_s(0)p) + (1-q)\tfrac{1}{2} = q(1-p) + (1-q)\tfrac{1}{2}.\end{gathered}$$

Shareholder i's vote affects the decision if $\frac{N-1}{2}$ other shareholders vote "for" and $\frac{N-1}{2}$ vote "against." The expected value of the proposal to shareholder i in this case is

$$\tilde{u}(s_i) = \mathbb{E}[u(1,\theta)|s_i, PIV_i] = \Pr[\theta = 1|s_i, PIV_i] - \Pr[\theta = 0|s_i, PIV_i],$$

where PIV_i denotes the state in which shareholder i's vote determines the outcome (i.e., if $\sum_{i \neq j} v_j = \frac{N-1}{2}$). Applying the Bayes rule,

$$\begin{aligned}\tilde{u}(s_i) &= \frac{\Pr[s_i|\theta=1]\Pr\left[\sum_{j\neq i} v_j = \frac{N-1}{2}|\theta=1\right] - \Pr[s_i|\theta=0]\Pr\left[\sum_{j\neq i} v_j = \frac{N-1}{2}|\theta=0\right]}{\Pr[s_i|\theta=1]\Pr\left[\sum_{j\neq i} v_j = \frac{N-1}{2}|\theta=1\right] + \Pr[s_i|\theta=0]\Pr\left[\sum_{j\neq i} v_j = \frac{N-1}{2}|\theta=0\right]} \\ &= D(s_i) \times \begin{pmatrix} \Pr[s_i|\theta=1]\left(qp + (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}}\left(1 - qp - (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}} \\ -\Pr[s_i|\theta=0]\left(q(1-p) + (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}}\left(1 - q(1-p) - (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}} \end{pmatrix} \\ &= D(s_i) \times (\Pr[s_i|\theta=1] - \Pr[s_i|\theta=0])\left(\tfrac{1}{2} + q\left(p - \tfrac{1}{2}\right)\right)^{\frac{N-1}{2}}\left(\tfrac{1}{2} - q\left(p - \tfrac{1}{2}\right)\right)^{\frac{N-1}{2}},\end{aligned}$$

where $D(s_i) > 0$. The best response of shareholder i is to vote "for" ($v_i = 1$) if $\tilde{u}(s_i) \geq 0$ and vote "against" ($v_i = 0$) if $\tilde{u}(s_i) \leq 0$. When $s_i = 1$, $\Pr[s_i|\theta = 1] - \Pr[s_i|\theta = 0] = 2p - 1 > 0$. When

$s_i = 0$, $\Pr[s_i|\theta = 1] - \Pr[s_i|\theta = 0] = 1 - 2p < 0$. Therefore, the optimal strategy of shareholder i is indeed $v_i = s_i$. Hence, $w_s(s) = s$ is an equilibrium.

Similarly, for an uninformed shareholder, the expected value of the proposal conditinal on being pivotal is

$$\tilde{u}_0 = D_0 \times \left(\begin{array}{c} \left(qp + (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}}\left(1 - qp - (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}} \\ -\left(q(1-p) + (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}}\left(1 - q(1-p) - (1-q)\frac{1}{2}\right)^{\frac{N-1}{2}} \end{array} \right) = 0,$$

for some D_0, and hence it is indeed optimal to mix between voting "for" and "against."

Next, consider shareholder i's value from becoming informed. Conditional on the shareholder's private signal being $s_i = 1$, whether he is informed or not only makes a difference if the number of "for" votes among other shareholders is exactly $\frac{N-1}{2}$. Let us denote this set of events by PIV_i. In this case, by acquiring the signal, the shareholder votes "for" for sure, instead of randomizing between voting "for" and "against," so his utility from being informed is $\frac{1}{2}\mathbb{E}[u(1,\theta)|s_i = 1, PIV_i]$. Similarly, conditional on his private signal being $s_i = 0$, the shareholder's utility from being informed is $-\frac{1}{2}\mathbb{E}[u(1,\theta)|s_i = 0, PIV_i]$. Overall, the shareholder's value of acquiring a private signal is

$$\begin{aligned} V(q) = &\Pr(s_i = 1)\Pr(PIV_i|s_i = 1)\tfrac{1}{2}\mathbb{E}[u(1,\theta)|s_i = 1, PIV_i] \\ &- \Pr(s_i = 0)\Pr(PIV_i|s_i = 0)\tfrac{1}{2}\mathbb{E}[u(1,\theta)|s_i = 0, PIV_i]. \end{aligned}$$

By the symmetry of the setup and strategies, $\mathbb{E}[u(1,\theta)|s_i = 1, PIV_i] = -\mathbb{E}[u(1,\theta)|s_i = 0, PIV_i]$ and $\Pr(PIV_i|s_i = 1) = \Pr(PIV_i|s_i = 0)$, so we get

$$\begin{aligned} V(q) &= \tfrac{1}{2}\Pr(PIV_i|s_i = 1)\,\mathbb{E}[u(1,\theta)|s_i = 1, PIV_i] \\ &= \tfrac{1}{2}\Pr(PIV_i|s_i = 1)\left(\Pr[\theta = 1|s_i = 1, PIV_i] - \Pr[\theta = 0|s_i = 1, PIV_i]\right) \\ &= \Pr[\theta = 1, PIV_i, s_i = 1] - \Pr[\theta = 0, PIV_i, s_i = 1] = \tfrac{1}{2}p\Pr[PIV_i|\theta = 1] - \tfrac{1}{2}(1-p)\Pr[PIV_i|\theta = 0] \end{aligned}$$

Conditional on $\theta = 1$, other shareholders make their voting decisions independently and vote "for" with probability $qp + \frac{1}{2}(1-q) = \frac{1}{2} + q\left(p - \frac{1}{2}\right)$. Hence,

$$\begin{aligned} \Pr[PIV_i|\theta = 1] &= P\left(\tfrac{1}{2} + q(p - \tfrac{1}{2}), N-1, \tfrac{N-1}{2}\right) \\ &= C_{N-1}^{\frac{N-1}{2}}\left(\tfrac{1}{2} + q(p - \tfrac{1}{2})\right)^{\frac{N-1}{2}}\left(\tfrac{1}{2} - q(p - \tfrac{1}{2})\right)^{\frac{N-1}{2}}. \end{aligned}$$

Noting that $\Pr[PIV_i|\theta = 1] = \Pr[PIV_i|\theta = 0]$, gives (5). Note that $V(q)$ decreases in q. Since $P\left(x, N-1, \frac{N-1}{2}\right)$ decreases in N for any x, it follows that $V(q)$ decreases in N. Finally, $V(q)$ increases in p if and only if

$$\begin{aligned} &\left[\hat{p}\left(\tfrac{1}{4} - q^2\hat{p}^2\right)^{\frac{N-1}{2}}\right]' > 0 \Leftrightarrow \left(\tfrac{1}{4} - q^2\hat{p}^2\right)^{\frac{N-1}{2}} - 2q^2\hat{p}\hat{p}\tfrac{N-1}{2}\left(\tfrac{1}{4} - q^2\hat{p}^2\right)^{\frac{N-1}{2}-1} \\ &= \left(\tfrac{1}{4} - q^2\hat{p}^2\right)^{\frac{N-1}{2}-1}\left[\tfrac{1}{4} - Nq^2\hat{p}^2\right] > 0 \Leftrightarrow 4Nq^2\left(p - \tfrac{1}{2}\right)^2 < 1, \end{aligned}$$

where $\hat{p} = p - \frac{1}{2}$. Hence, the value of acquiring information decreases in the precision p if N, q, and p are large enough. Intuitively, in this case, many shareholders become informed and vote according to their precise private signals, so the probability that their votes are split is not very high. That reduces the shareholder's ability to affect the decision with his vote, reducing the value of his private information.

In deciding whether to acquire the private signal, shareholder i compares the expected value of his signal $V(q)$ with cost c. A shareholder acquires the signal if and only if the former exceeds the latter. Since the value of shareholder i's information is strictly decreasing in the fraction of other shareholders acquiring information, there exists a unique q that solves the equation $V(q) = c$, unless c is very low or very high. If c is very low or very high, then either all shareholders acquire information or none of them do. Specifically, note that $V(1) = \underline{c}$ and $V(0) = \bar{c}$, where $\underline{c}$ and $\bar{c}$ are given by (6). Since $V(q)$ is strictly decreasing in q, if $c < \underline{c} \equiv V(1)$, then each shareholder acquires information regardless of q. Hence, in the unique symmetric equilibrium all shareholders acquire private signals: $q^* = 1$. If $c > \bar{c} \equiv V(0)$, then each shareholder is better off not acquiring information regardless of q. Hence, in the unique symmetric equilibrium all shareholders remain uninformed: $q^* = 0$. Finally, if $c \in [\underline{c}, \bar{c}]$, then q^* is given as the solution to $V(q^*) = c$. Plugging (5) and rearranging the terms, we get (6).

Finally, we derive the equilibrium firm value given q_0^* :

$$V_0 = \Pr(\theta = 1) \sum_{k=\frac{N+1}{2}}^{N} P(q_0^* p + \tfrac{1-q_0^*}{2}, N, k) - \Pr(\theta = 0) \sum_{k=\frac{N+1}{2}}^{N} P(q_0^*(1-p) + \tfrac{1-q_0^*}{2}, N, k)$$
$$= \tfrac{1}{2} \sum_{k=\frac{N+1}{2}}^{N} [P(\tfrac{1}{2} + \Lambda, N, k) - P(\tfrac{1}{2} - \Lambda, N, k)]$$
$$= \tfrac{1}{2} \sum_{k=\frac{N+1}{2}}^{N} P(\tfrac{1}{2} + \Lambda, N, k) - \tfrac{1}{2} \sum_{k=\frac{N+1}{2}}^{N} P(\tfrac{1}{2} + \Lambda, N, N-k)]$$
$$= \tfrac{1}{2} \sum_{k=\frac{N+1}{2}}^{N} P(\tfrac{1}{2} + \Lambda, N, k) - \tfrac{1}{2} \sum_{m=0}^{\frac{N-1}{2}} P(\tfrac{1}{2} + \Lambda, N, m)] = \sum_{k=\frac{N+1}{2}}^{N} P(\tfrac{1}{2} + \Lambda, N, k) - \tfrac{1}{2}.$$

Proof of Proposition 2.

We start by showing that there is no symmetric equilibrium in which a shareholder acquires both signals with a positive probability. Suppose such an equilibrium exists and consider a shareholder who acquired both signals, r and s_i. Suppose the two signals disagree, $r = 1$, $s_i = 0$. There are three possible cases: $w_{rs}(1,0) = 1$, $w_{rs}(1,0) = 0$, and $w_{rs}(1,0) \in (0,1)$.

First, if $w_{rs}(1,0) = 1$, then it must be that $w_{rs}(1,1) = 1$ because the shareholder is more optimistic when both signals are positive than when one of them is negative. The symmetry assumption implies that $w_{rs}(0,1) = 1 - w_{rs}(1,0) = 0$, and hence if $r = 0$, $s_i = 1$, the shareholder must vote against. Then $w_{rs}(0,0) = 0$ because the shareholder is more pessimistic when both signals are negative than when one of them is positive. Overall, it follows that $v_i = r$, and hence the shareholder would be better off if he did not pay for the private signal and always followed the advisor's signal.

Second, if $w_{rs}(1,0) = 0$, then it must be that $w_{rs}(0,0) = 0$. The symmetry assumption implies that $w_{rs}(0,1) = 1 - w_{rs}(1,0) = 1$, and hence $w_{rs}(1,1) = 1$. Overall, it follows that $v_i = s_i$, and hence the shareholder would be better off if he did not pay for the proxy advisor's signal and always followed his private signal.

Finally, if $w_{rs}(1,0) \in (0,1)$, the shareholder must be indifferent between voting for and against. It follows that he strictly prefers voting according to the signals when they agree with each other: $w_{rs}(1,1) = 1$ and $w_{sr}(0,0) = 0$. In addition, the symmetry assumption implies that $w_{rs}(0,1) \in (0,1)$, i.e., the shareholder must be indifferent whenever the two signals disagree. This, in turn, implies that his utility from voting would be exactly the same if he always voted according to the advisor's signal. Hence, he would be strictly better off not paying for his private signal. Similarly, he would be strictly better off if he did not acquire the advisor's signal and always followed his private signal.

Hence, we can focus on equilibria in which each shareholder acquires the signal of the advisor with probability q_r, acquires a private signal with probability q_s, and stays uninformed with probability $q_n = 1 - q_r - q_s$. Such an equilibrium only exists if given q_r, q_s, it is optimal for a shareholder who acquired a signal to follow it. In the text, we show that given the symmetry of the game, it is always optimal for a sharehoder who acquired a private signal to follow it. Next, we analyze the decision of a shareholder who acquired the advisor's signal. Equilibrium with $q_r > 0$ will only exist if such a shareholder finds it optimal to follow the advisor's signal.

It will be useful to compute the probabilities that a random shareholder j votes for the proposal, conditional on the advisor's recommendation r and the true state θ:

$$\Pr[v_j = 1|r = 1, \theta = 1] = q_r + q_s p + q_n \frac{1}{2} \tag{17}$$

$$\Pr[v_j = 1|r = 0, \theta = 1] = q_s p + q_n \frac{1}{2} \tag{18}$$

$$\Pr[v_j = 1|r = 1, \theta = 0] = q_r + q_s(1-p) + q_n \frac{1}{2} \tag{19}$$

$$\Pr[v_j = 1|r = 0, \theta = 0] = q_s(1-p) + q_n \frac{1}{2} \tag{20}$$

Intuitively, if shareholder j buys the advisor's recommendation, which occurs with probability q_r, he votes for the proposal if and only if the advisor recommends it (i.e., if $r = 1$). If shareholder j acquires a private signal s_j, which occurs with probability q_s, he votes for the proposal if and only if his private signal is $s_j = 1$, which occurs with probability p if the state is $\theta = 1$, or with probability $1 - p$ if the state is $\theta = 0$. Finally, if shareholder j stays uninformed, which occurs with probability q_n, he votes for the proposal with probability 50%, regardless of the state θ and the advisor's recommendation r.

Consider (**??**). Using Bayes' rule and the fact that $\Pr(\theta) = \frac{1}{2} = \Pr(r)$, we get

$$\begin{gathered} e_r(r, q_s, q_r) = \mathbb{E}[u(1,\theta)|r, PIV_i] \Pr[PIV_i|r] \\ = \Pr[\theta = 1|r, PIV_i] \Pr[PIV_i|r] - \Pr[\theta = 0|r, PIV_i] \Pr[PIV_i|r] \\ = \Pr[\theta = 1, PIV_i|r] - \Pr[\theta = 0, PIV_i|r] \\ = 2(\Pr[\theta = 1, PIV_i, r] - \Pr[\theta = 0, PIV_i, r]) \\ = \Pr[PIV_i|r, \theta = 1] \Pr[r|\theta = 1] - \Pr[PIV_i|r, \theta = 0] \Pr[r|\theta = 0] \end{gathered} \tag{21}$$

By conditional independence of s_i, s_j, $j \neq i$, and r conditional on θ,

$$\Pr[PIV_i|r, \theta] = \Pr\left[\sum_{j \neq i} v_j = \frac{N-1}{2}|r, \theta\right] = P\left(\Pr[v_j = 1|r, \theta], N-1, \frac{N-1}{2}\right).$$

Plugging this into (21) gives (**??**).

For such an equilibrium to exist, the shareholder with recommendation $r = 1$ ($r = 0$) must find it optimal to vote for (against) the proposal, which requires $e_r(1, q_s, q_r) \geq 0$ and $e_r(0, q_s, q_r) \leq 0$. By the symmetry of the problem, $-e_r(0, q_s, q_r) = e_r(1, q_s, q_r)$. Therefore, it is sufficient to verify that $e_r(1, q_s, q_r) \geq 0$. Plugging (17) and (19) into (**??**), this is the case if and only if (8) holds.

Denote the function in the brackets of (8) by $f(q_r, q_s, p)$. Note that

$$f = \frac{\frac{1}{4} - \left(\frac{q_r}{2} + q_s\left(p - \frac{1}{2}\right)\right)^2}{\frac{1}{4} - \left(\frac{q_r}{2} - q_s\left(p - \frac{1}{2}\right)\right)^2} = \frac{\left(q_r + q_s(2p-1)\right)^2 - 1}{\left(q_r - q_s(2p-1)\right)^2 - 1} = \frac{\left(q_s(2p-1) + q_r\right)^2 - 1}{\left(q_s(2p-1) - q_r\right)^2 - 1}.$$

Consider the function $f = \frac{(a+b)^2-1}{(a-b)^2-1}$ and note that

$$\begin{aligned} f'_a &< 0 \Leftrightarrow (a+b)\left(a^2+b^2-2ab-1\right) < \left(a^2+b^2+2ab-1\right)(a-b) \\ &\Leftrightarrow a^3+ab^2-2a^2b-a+a^2b+b^3-2ab^2-b < a^3+ab^2+2a^2b-a-a^2b-b^3-2ab^2+b \\ &\Leftrightarrow 2b^3 < 2a^2b+2b \Leftrightarrow b^2 < a^2+1. \end{aligned}$$

Hence, if $a = q_s(2p-1), b = q_r$, we have $q_r^2 < q_s^2(2p-1)^2 + 1$ because $q_r \leq 1$. Similarly, if $a = q_r, b = q_s(2p-1)$, we have $q_s^2(2p-1)^2 < q_r^2 + 1$ because $q_s(2p-1) \leq 1$. Hence, f decreases in q_r, q_s, p.

Value of signals. We derive the value of the private signal $V_s(q_r, q_s)$ and the value of the advisor's recommendation $V_r(q_r, q_s)$ to shareholder i for given q_r, q_s.

1. Value of private signal. Consider shareholder i's value from acquiring a private signal. Conditional on the shareholder's private signal being $s_i = 1$, whether he is informed or not only makes a difference if the number of "for" votes among other shareholders is exactly $\frac{N-1}{2}$. Let us denote this event by PIV_i. In this case, by acquiring the signal, the shareholder votes "for" for sure, instead of randomizing between voting "for" and "against," so his utility from being informed is $\frac{1}{2}\mathbb{E}\left[u(1,\theta)\,|s_i = 1, PIV_i\right]$. Similarly, conditional on his private signal being $s_i = 0$, the shareholder's utility from being informed is $-\frac{1}{2}\mathbb{E}\left[u(1,\theta)\,|s_i = 0, PIV_i\right]$. Overall, the shareholder's value of acquiring a private signal is

$$\begin{gathered} V_s(q_r, q_s) = \Pr(s_i = 1)\Pr(PIV_i|s_i = 1)\,\tfrac{1}{2}\mathbb{E}\left[u(1,\theta)\,|s_i = 1, PIV_i\right] \\ - \Pr(s_i = 0)\Pr(PIV_i|s_i = 0)\,\tfrac{1}{2}\mathbb{E}\left[u(1,\theta)\,|s_i = 0, PIV_i\right]. \end{gathered}$$

By the symmetry of the setup and strategies, $\mathbb{E}\left[u(1,\theta)\,|s_i = 1, PIV_i\right] = -\mathbb{E}\left[u(1,\theta)\,|s_i = 0, PIV_i\right]$ and $\Pr(PIV_i|s_i = 1) = \Pr(PIV_i|s_i = 0)$, so we get

$$\begin{gathered} V_s(q_r, q_s) = \tfrac{1}{2}\Pr(PIV_i|s_i = 1)\,\mathbb{E}\left[u(1,\theta)\,|s_i = 1, PIV_i\right] \\ = \tfrac{1}{2}\Pr(PIV_i|s_i = 1)\left(\Pr\left[\theta = 1|s_i = 1, PIV_i\right] - \Pr\left[\theta = 0|s_i = 1, PIV_i\right]\right) \\ = \tfrac{1}{2}\Pr(PIV_i, \theta = 1|s_i = 1) - \tfrac{1}{2}\Pr(PIV_i, \theta = 0|s_i = 1) \\ = \tfrac{1}{2}\Pr(PIV_i, s_i = 1|\theta = 1) - \tfrac{1}{2}\Pr(PIV_i, s_i = 1|\theta = 0) \\ = \tfrac{1}{2}\left(p\Pr(PIV_i|\theta = 1) - (1-p)\Pr(PIV_i|\theta = 0)\right) = \left(p - \tfrac{1}{2}\right)\Pr(PIV_i|\theta = 1). \end{gathered}$$

Note that

$$\begin{gathered} \Pr(PIV_i|\theta = 1) = \pi\Pr(PIV_i|r = 1, \theta = 1) + (1-\pi)\Pr(PIV_i|r = 0, \theta = 1) \\ = \pi P\left(\tfrac{1}{2}q_u + q_r + q_s p, N-1, \tfrac{N-1}{2}\right) \\ + (1-\pi)P\left(\tfrac{1}{2}q_u - q_r + q_s p, N-1, \tfrac{N-1}{2}\right). \end{gathered}$$

Hence, $V_s(q_r, q_s)$ is given by (9).

2. Value of the advisor's signal. Consider shareholder i's value from acquiring the advisor's signal. Conditional on the advisor's signal being $r = 1$, whether the shareholder knows it or not only makes a difference if the number of "for" votes among other shareholders is exactly $\frac{N-1}{2}$. Let us denote this set of events by PIV_i. In this case, by acquiring the signal, the shareholder votes "for" for sure, instead of randomizing between voting "for" and "against," so his utility from being informed is $\frac{1}{2}\mathbb{E}[u(1,\theta)|r = 1, PIV_i]$. Similarly, conditional on the advisor's signal being $r = 0$, the shareholder's utility from being informed is $-\frac{1}{2}\mathbb{E}[u(1,\theta)|r = 0, PIV_i]$. Overall, the shareholder's value of acquiring the advisor's signal is

$$\begin{aligned} V_r(q_r, q_s) = &\Pr(r=1)\Pr(PIV_i|r=1)\tfrac{1}{2}\mathbb{E}[u(1,\theta)|r=1, PIV_i] \\ &-\Pr(r=0)\Pr(PIV_i|r=0)\tfrac{1}{2}\mathbb{E}[u(1,\theta)|r=0, PIV_i]. \end{aligned}$$

By the symmetry of the setup and strategies, $\mathbb{E}[u(1,\theta)|r = 1, PIV_i] = -\mathbb{E}[u(1,\theta)|r = 0, PIV_i]$ and $\Pr(PIV_i|r = 1) = \Pr(PIV_i|r = 0)$, so we get

$$\begin{aligned} V_r(q_r, q_s) &= \tfrac{1}{2}\Pr(PIV_i|r=1)\,\mathbb{E}[u(1,\theta)|r=1, PIV_i] \\ &= \tfrac{1}{2}\Pr(PIV_i|r=1)\left(\Pr(\theta=1|r=1, PIV_i) - \Pr(\theta=0|r=1, PIV_i)\right) \\ &= \tfrac{1}{2}\left(\Pr(PIV_i, \theta=1|r=1) - \Pr(PIV_i, \theta=0|r=1)\right) \\ &= \Pr(\theta=1, PIV_i, r=1) - \Pr(\theta=0, PIV_i, r=1) \\ &= \tfrac{1}{2}\Pr(PIV_i|r=1,\theta=1)\Pr(r=1|\theta=1) - \tfrac{1}{2}\Pr(PIV_i|r=1,\theta=0)\Pr(r=1|\theta=0) \\ &= \tfrac{1}{2}\Pr(PIV_i|r=1,\theta=1)\,\pi - \tfrac{1}{2}\Pr(PIV_i|r=1,\theta=0)(1-\pi). \end{aligned}$$

Note that

$$\begin{aligned} \Pr(PIV_i|r=1,\theta=1) &= P\left(q_r + q_s p + \tfrac{1}{2}q_u, N-1, \tfrac{N-1}{2}\right) \\ \Pr(PIV_i|r=1,\theta=0) &= P\left(q_r - q_s p + \tfrac{1}{2}q_u, N-1, \tfrac{N-1}{2}\right) \end{aligned}$$

Hence, $V_r(q_r, q_s)$ is given by (10).

Proof of Lemma 2. We prove the lemma in steps. First, we derive the necessary and sufficient conditions for each type of equilibrium to exist. We start with the equilibrium with complete crowding out of private information acquisition and follow with the equilibrium with incomplete crowding out of private information acquisition. Then, we prove the result about the ranking of equilibria in shareholder welfare.

1. Equilibrium with complete crowding out. First, consider the equilibrium with complete crowding out of private information acquisition, $q_s = 0$ and $q_r \in (0, 1)$ (note that q_r cannot be equal to one because otherwise, no shareholder is pivotal, and hence there is no benefit from becoming informed). The following conditions must be satisfied: $V_s(q_r, 0) \leq c$ and $V_r(q_r, 0) = f$.When $q_s = 0$, the probabilities of being pivotal are:

$$\Omega_1(q_r, 0) = P\left(\frac{1+q_r}{2}, N-1, \frac{N-1}{2}\right) = P\left(\frac{1-q_r}{2}, N-1, \frac{N-1}{2}\right) = \Omega_2(q_r, 0) = \Omega_r(q_r, 0). \quad (22)$$

Equation $V_r(q_r,0)=f$ yields $\Omega_r(q_r)=\frac{2f}{2\pi-1}$. Plugging (22) into it, we get

$$q_r=\sqrt{1-4\left(\frac{2f}{C_{N-1}^{\frac{N-1}{2}}(2\pi-1)}\right)^{\frac{2}{N-1}}}. \tag{23}$$

For this case to hold, first, it must be the case that

$$\left(\frac{2f}{(2\pi-1)\,C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}\leq\frac{1}{4}\Leftrightarrow f\leq C_{N-1}^{\frac{N-1}{2}}2^{1-N}\left(\pi-\frac{1}{2}\right),$$

as otherwise no one has incentives to buy the recommendation from the advisor. Second, it must be the case that $V_s(q_r,0)\leq c$, i.e., given that fraction q_r buys from the proxy advisor, no one wants to acquire private information, which requires

$$(p-\frac{1}{2})\Omega_r(q_r)\leq c.$$

Plugging $\Omega_r(q_r)=\frac{2f}{2\pi-1}$, we obtain $f\leq\frac{2\pi-1}{2p-1}c$. To sum up, the equilibrium with complete crowding out of private information acquisition exists if and only if $f\leq\min\left\{C_{N-1}^{\frac{N-1}{2}}2^{1-N}\left(\pi-\frac{1}{2}\right),\frac{2\pi-1}{2p-1}c\right\}$. Note that

$$C_{N-1}^{\frac{N-1}{2}}2^{1-N}(\pi-\frac{1}{2})\;>\;\frac{2\pi-1}{2p-1}c\Leftrightarrow\frac{1}{4}>\left(\frac{2c}{(2p-1)\,C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}},$$

which is satisfied by Assumption 1. Hence, the equilibrium with complete crowding out of information exists if and only if $f\leq\frac{2\pi-1}{2p-1}c\equiv\bar{f}$.

2. Equilibrium with incomplete crowding out. Second, consider the equilibrium with incomplete crowding out of private information acquisition. When $q_r+q_s<1$, the equilibrium must satisfy $V_s(q_r,q_s)=c$ and $V_r(q_r,q_s)=f$,which yields a system of linear equations determining Ω_1 and Ω_2:

$$\begin{cases}\pi\Omega_1+(1-\pi)\Omega_2=\frac{2c}{2p-1}\\ \pi\Omega_1-(1-\pi)\Omega_2=2f\end{cases}\Leftrightarrow\Omega_1=\frac{f+\frac{c}{2p-1}}{\pi}\text{ and }\Omega_2=\frac{\frac{c}{2p-1}-f}{1-\pi}. \tag{24}$$

The solution for Ω_1 and Ω_2 corresponds to the following system of equations for q_r and q_s:

$$\begin{aligned}C_{N-1}^{\frac{N-1}{2}}\left(\left(\tfrac{1}{2}+\tfrac{1}{2}q_r+\left(p-\tfrac{1}{2}\right)q_s\right)\left(\tfrac{1}{2}-\tfrac{1}{2}q_r-\left(p-\tfrac{1}{2}\right)q_s\right)\right)^{\frac{N-1}{2}}&=\frac{f+\frac{c}{2p-1}}{\pi},\\ C_{N-1}^{\frac{N-1}{2}}\left(\left(\tfrac{1}{2}-\tfrac{1}{2}q_r+\left(p-\tfrac{1}{2}\right)q_s\right)\left(\tfrac{1}{2}+\tfrac{1}{2}q_r-\left(p-\tfrac{1}{2}\right)q_s\right)\right)^{\frac{N-1}{2}}&=\frac{\frac{c}{2p-1}-f}{1-\pi}.\end{aligned}$$

If $f>\frac{c}{2p-1}$, then the right-hand side of the second equation is negative, and since the left-hand side is non-negative, no solution exists. Thus, a necessary condition for this system to have a solution is $f \leq \frac{c}{2p-1}$. In this case,

$$\left(\tfrac{1}{2}q_r + \left(p - \tfrac{1}{2}\right) q_s\right)^2 = \tfrac{1}{4} - \left(\frac{f + \frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}},$$
$$\left(\tfrac{1}{2}q_r - \left(p - \tfrac{1}{2}\right) q_s\right)^2 = \tfrac{1}{4} - \left(\frac{\frac{c}{2p-1} - f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}.$$

Because the left-hand side of the first equation is strictly greater than the left-hand side of the second equation for $q_s > 0$, the same must be true about the right-hand sides, and hence $\frac{f + \frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}} < \frac{\frac{c}{2p-1} - f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}} \Leftrightarrow \Omega_1 < \Omega_2$ is the necessary condition for the solution to exist. Note that $\Omega_1 < \Omega_2$ if and only if $f < \frac{2\pi-1}{2p-1}c = \bar{f}$. Since $2\pi - 1 < 1$, the condition $f \leq \frac{c}{2p-1}$ is automatically satisfied if $f < \bar{f}$.

It also follows that two other necessary conditions for existence of the solution is that the right-hand sides of the two equations above are positive, or equivalently,

$$\frac{f + \frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}} < 2^{1-N} \Leftrightarrow f < 2^{1-N}\pi C_{N-1}^{\frac{N-1}{2}} - \tfrac{c}{2p-1},$$
$$\frac{\frac{c}{2p-1} - f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}} \leq 2^{1-N} \Leftrightarrow f \geq \tfrac{c}{2p-1} - 2^{1-N}(1-\pi) C_{N-1}^{\frac{N-1}{2}} \equiv \underline{f}.$$

Note that

$$2^{1-N}\pi C_{N-1}^{\frac{N-1}{2}} - \frac{c}{2p-1} \; > \; \frac{2\pi - 1}{2p-1}c = \bar{f} \Leftrightarrow \frac{1}{4} \; > \; \left(\frac{2c}{(2p-1)\, C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}},$$

which is satisfied by Assumption 1. Hence, the first inequality follows from the inequality $f < \frac{2\pi-1}{2p-1}c = \bar{f}$, which must be satisfied in this equilibrium. Overall, the necessary conditions for existence of this equilibrium are $f < \frac{2\pi-1}{2p-1}c = \bar{f}$ and $f \geq \underline{f}$.

Under $\underline{f} \leq f < \bar{f}$, the system is solved by:

$$\begin{aligned} \tfrac{1}{2}q_r + \left(p - \tfrac{1}{2}\right) q_s &= \sqrt{\tfrac{1}{4} - \left(\frac{f + \frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} \\ \tfrac{1}{2}q_r - \left(p - \tfrac{1}{2}\right) q_s &= \pm\sqrt{\tfrac{1}{4} - \left(\frac{\frac{c}{2p-1} - f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} \end{aligned} \qquad (25)$$

First, when $f = \underline{f}$, the right-hand side of the second equation is zero, and hence the system has a unique solution, which satisfies

$$q_r = (2p-1)\, q_s = \sqrt{\frac{1}{4} - \left(\frac{\underline{f} + \frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} = \sqrt{\frac{1}{4} - \left(\frac{\frac{2c}{2p-1} - 2^{1-N}(1-\pi)\, C_{N-1}^{\frac{N-1}{2}}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}.$$

Second, when $f > \underline{f}$, the system has two solutions:

1. **Solution 1:** $q_r > (2p-1)\, q_s$. In this case, the solution to (25) is given by

$$\begin{aligned} q_r &= \sqrt{\tfrac{1}{4} - \left(\frac{f+\frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} + \sqrt{\tfrac{1}{4} - \left(\frac{\frac{c}{2p-1}-f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} \\ q_s &= \tfrac{1}{2p-1}\left(\sqrt{\tfrac{1}{4} - \left(\frac{f+\frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} - \sqrt{\tfrac{1}{4} - \left(\frac{\frac{c}{2p-1}-f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}\right) \end{aligned}$$

2. **Solution 2:** $q_r < (2p-1)\, q_s$. In this case, the solution to (25) is given by

$$\begin{aligned} q_r &= \sqrt{\tfrac{1}{4} - \left(\frac{f+\frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} - \sqrt{\tfrac{1}{4} - \left(\frac{\frac{c}{2p-1}-f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} \\ q_s &= \tfrac{1}{2p-1}\left(\sqrt{\tfrac{1}{4} - \left(\frac{f+\frac{c}{2p-1}}{\pi C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}} + \sqrt{\tfrac{1}{4} - \left(\frac{\frac{c}{2p-1}-f}{(1-\pi) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}\right) \end{aligned} \tag{26}$$

Whenever $q_r + q_s > 1$ for one or both of these solutions, the solution does not correspond to an equilibrium. In this case, the equilibrium has $q_r + q_s = 1$, and q_r and q_s satisfy $V_s(q_r, q_s) - c = V_r(q_r, q_s) - f$. Therefore,

$$\left(p - \frac{1}{2}\right)(\pi\Omega_1 + (1-\pi)\,\Omega_2) - c = \frac{1}{2}(\pi\Omega_1 - (1-\pi)\,\Omega_2) - f \equiv \Psi > 0.$$

Let $\hat{c} \equiv c + \Psi$ and $\hat{f} \equiv f + \Psi$. Then, we have two solutions for q_r and q_s, which are identical to the ones above, but with $\hat{c}$ and $\hat{f}$ instead of c and f.

Finally, if $f > \frac{2\pi-1}{2p-1} c = \bar{f}$, then neither of these equilibria exist, since each shareholder strictly prefers acquiring private information over buying the advisor's recommendation. Thus, the equilibrium is identical to the benchmark case. By the same argument as before, there exists a unique symmetric equilibrium in this case.

3. Ranking of equilibria in shareholder welfare when $f \in (\underline{f}, \bar{f})$. Consider an equilibrium defined by pair q_s and q_r. Let $U(q_s, q_r)$ denote the expected value of a proposal per share in it. By definition,

$$
\begin{aligned}
U\left(q_s, q_r\right) = \mathbb{E}\left[u\left(1,\theta\right)d\right] &= \tfrac{1}{2}\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=1\right] - \tfrac{1}{2}\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=0\right] \\
&= \tfrac{1}{2}\left(\pi\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=r=1\right] + \left(1-\pi\right)\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=1, r=0\right]\right) \\
&\quad -\tfrac{1}{2}\left(\pi\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=r=0\right] + \left(1-\pi\right)\mathbb{E}\left[\sum v_j > \tfrac{N-1}{2}|\theta=0, r=1\right]\right) \\
&= \tfrac{1}{2}\pi\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_a, N, k\right) - \sum_{k=\frac{N+1}{2}}^{N} P\left(1-p_a, N, k\right)\right) \\
&\quad +\tfrac{1}{2}\left(1-\pi\right)\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_d, N, k\right) - \sum_{k=\frac{N+1}{2}}^{N} P\left(1-p_d, N, k\right)\right).
\end{aligned}
$$

where

$$
\begin{aligned}
p_a &= \tfrac{1}{2} + \tfrac{1}{2}q_r + \left(p-\tfrac{1}{2}\right)q_s, \\
p_d &= \tfrac{1}{2} - \tfrac{1}{2}q_r + \left(p-\tfrac{1}{2}\right)q_s.
\end{aligned}
$$

Since $P\left(q, N, k\right) = P\left(1-q, N, N-k\right)$,

$$
\begin{aligned}
U\left(q_s, q_r\right) &= \tfrac{1}{2}\pi\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_a, N, k\right) - \sum_{k=0}^{\frac{N-1}{2}} P\left(p_a, N, k\right)\right) \\
&\quad +\tfrac{1}{2}\left(1-\pi\right)\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_d, N, k\right) - \sum_{k=0}^{\frac{N-1}{2}} P\left(p_d, N, k\right)\right) \\
&= \sum_{k=\frac{N+1}{2}}^{N}\left(\pi P\left(p_a, N, k\right) + \left(1-\pi\right)P\left(p_d, N, k\right)\right) - \tfrac{1}{2},
\end{aligned}
\tag{27}
$$

where the last equality follows from $\sum_{k=0}^{\frac{N-1}{2}} P\left(q, N, k\right) = 1 - \sum_{k=\frac{N+1}{2}}^{N} P\left(q, N, k\right)$. The expected welfare of a shareholder is the expected value of a proposal per share, $U\left(q_s, q_r\right)$, minus the expected information acquisition cost:

$$
W\left(q_s, q_r\right) = \sum_{k=\frac{N+1}{2}}^{N}\left(\pi P\left(p_a, N, k\right) + \left(1-\pi\right)P\left(p_d, N, k\right)\right) - \frac{1}{2} - q_r f - q_s c. \tag{28}
$$

First, we show that the equilibrium with incomplete crowding out of private information acquisition and $q_r > \left(2p-1\right)q_s$, denoted $(q_s^{(2)}, q_r^{(2)})$ has lower shareholder welfare than the equilibrium with incomplete crowding out of private information acquisition and $q_r < \left(2p-1\right)q_s$, denoted $(q_s^{(1)}, q_r^{(1)})$. Without loss of generality, suppose that $q_r + q_s < 1$. If $q_r + q_s = 1$, the proof is identical with the replacement of c and f with $\hat{c}$ and $\hat{f}$. Plugging $q_r = p_a - p_d$ and $q_s = \frac{p_a+p_d-1}{2p-1}$ into (28), $W\left(q_s, q_r\right)$ can be rewritten as

$$
\sum_{k=\frac{N+1}{2}}^{N}\left(\pi P\left(p_a, N, k\right) + \left(1-\pi\right)P\left(p_d, N, k\right)\right) - \left(f + \frac{c}{2p-1}\right)p_a - \left(\frac{c}{2p-1} - f\right)p_d - \frac{1}{2} + \frac{c}{2p-1}.
$$

Using (24),

$$
W\left(q_s, q_r\right) = \pi\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_a, N, k\right) - \Omega_1 p_a\right) + \left(1-\pi\right)\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(p_d, N, k\right) - \Omega_2 p_d\right) - \frac{1}{2} + \frac{c}{2p-1}.
$$

Since p_a, Ω_1, and Ω_2 are identical in both equilibria and $p_d(q_s^{(2)}, q_r^{(2)}) = 1 - p_d(q_s^{(1)}, q_r^{(1)})$, the comparison of $W(q_s^{(1)}, q_r^{(1)})$ and $W(q_s^{(2)}, q_r^{(2)})$ is equivalent to the comparison of

$$\sum_{k=\frac{N+1}{2}}^{N} P\left(p_d, N, k\right) - \Omega_2 p_d \ \vee \ \sum_{k=\frac{N+1}{2}}^{N} P\left(1-p_d, N, k\right) - \Omega_2\left(1-p_d\right),$$

which is equivalent to

$$\sum_{k=\frac{N+1}{2}}^{N} P\left(p_d, N, k\right) - \frac{1}{2} \vee \left(p_d - \frac{1}{2}\right) P\left(p_d, N-1, \frac{N-1}{2}\right),$$

where $p_d > \frac{1}{2}$. Denote the left-hand side and the right-hand side by $L\left(p_d\right)$ and $R\left(p_d\right)$, respectively. Note that $L\left(\frac{1}{2}\right) = R\left(\frac{1}{2}\right) = 0$. Differentiating the left-hand side,

$$L'\left(x\right) = \sum_{k=\frac{N+1}{2}}^{N} P_1\left(x, N, k\right) = -\sum_{k=0}^{\frac{N-1}{2}} P_1\left(x, N, k\right) = -\frac{1}{x\left(1-x\right)}\left(\sum_{k=0}^{\frac{N-1}{2}} P\left(x, N, k\right)\left(k - Nx\right)\right).$$

Hence,

$$\begin{gathered} x\left(1-x\right) L'\left(x\right) = -\textstyle\sum_{k=0}^{\frac{N-1}{2}} kP\left(x, N, k\right) + Nx\left(\textstyle\sum_{k=0}^{\frac{N-1}{2}} P\left(x, N, k\right)\right) \\ = Nx\left(I_{1-x}\left(\frac{N+1}{2}, \frac{N+1}{2}\right) - I_{1-x}\left(\frac{N+1}{2}, \frac{N-1}{2}\right)\right) = Nx\frac{(1-x)^{\frac{N+1}{2}} x^{\frac{N-1}{2}}}{\frac{N-1}{2} B\left(\frac{N+1}{2}, \frac{N-1}{2}\right)} = \frac{((1-x)x)^{\frac{N+1}{2}} N!}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!} \\ = Nx\left(1-x\right) P\left(x, N-1, \frac{N-1}{2}\right), \end{gathered}$$

where $I_x\left(a, b\right)$ is the regularized incomplete beta function and $B\left(a, b\right)$ is the beta function. Differentiating the right-hand side,

$$\begin{gathered} R'\left(x\right) = P_1\left(x, N-1, \frac{N-1}{2}\right)\left(x - \frac{1}{2}\right) + P\left(x, N-1, \frac{N-1}{2}\right) \\ = P\left(x, N-1, \frac{N-1}{2}\right)\left(\frac{\frac{N-1}{2} - (N-1)x}{x(1-x)}\left(x - \frac{1}{2}\right) + 1\right) = P\left(x, N-1, \frac{N-1}{2}\right)\left(1 - \frac{(N-1)\left(x-\frac{1}{2}\right)^2}{x(1-x)}\right) \\ < P\left(x, N-1, \frac{N-1}{2}\right) N = L'\left(x\right). \end{gathered}$$

Therefore, $L\left(x\right) > R\left(x\right)$ for any $x > \frac{1}{2}$. Hence, $W(q_s^{(1)}, q_r^{(1)}) > W(q_s^{(2)}, q_r^{(2)})$.

Second, we show that the equilibrium with complete crowding out of private information acquisition and $q_r > \left(2p-1\right) q_s$ has a higher shareholder welfare than the equilibrium with complete crowding out of private information acquisition, denoted $(0, q_r^{(3)})$. Define function $\varphi\left(x\right) \in \left(\frac{1}{2}, 1\right)$ by

$$\varphi\left(x\right) \equiv \frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{x}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}, \text{ so that } x = C_{N-1}^{\frac{N-1}{2}}\left(\varphi\left(x\right)\left(1-\varphi\left(x\right)\right)\right)^{\frac{N-1}{2}}. \tag{29}$$

Since $p_d < \frac{1}{2}$ in both of these equilibria, we have $p_a = \varphi\left(\Omega_1\right)$ and $p_d = 1 - \varphi\left(\Omega_2\right)$. Plugging these expressions for p_a and p_d we can re-write (28) as

$$
\begin{aligned}
&\textstyle\sum_{k=\frac{N+1}{2}}^{N} \left(\pi P\left(\varphi\left(\Omega_1\right), N, k\right) + (1-\pi) P\left(1-\varphi\left(\Omega_2\right), N, k\right)\right) - \frac{1}{2} - q_r f - q_s c \\
&= \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left(\pi P\left(\varphi\left(\Omega_1\right), N, k\right) - (1-\pi) P\left(\varphi\left(\Omega_2\right), N, k\right)\right) + \frac{1}{2} - \pi - q_r f - q_s c,
\end{aligned}
$$

where we used $\sum_{k=\frac{n+1}{2}}^{n} P(1-x, n, k) = \sum_{k=0}^{\frac{n=1}{2}} P(x, n, k)$ and $\sum_{k=0}^{n} P(x, n, k) = 1$ to get to the second line. Plugging $q_r = p_a - p_d$, and $q_s = \frac{p_a+p_d-1}{2p-1}$ into the expression, we can write shareholder welfare $W(q_s, q_r)$ as a function of Ω_1 and Ω_2:

$$
\hat{W}\left(\Omega_1, \Omega_2\right) = \pi \tilde{f}\left(\Omega_1\right) - (1-\pi) \tilde{f}\left(\Omega_2\right) + \frac{1}{2} - \pi, \tag{30}
$$

where

$$
\tilde{f}(x) \equiv \sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k) - x\left(\varphi(x) - \frac{1}{2}\right). \tag{31}
$$

Shareholder welfare in the equilibrium with complete crowding out of private information acquisition is given by $\hat{W}(\Omega_r, \Omega_r)$, where $\Omega_r = \frac{2f}{2\pi-1}$. Similarly, shareholder welfare in the equilibrium with incomplete crowding of private information acquisition and $q_r > (2p-1) q_s$ is given by $\hat{W}(\Omega_1, \Omega_2)$ with Ω_1 and Ω_2 given by (24). Alternatively, we can write them as $\Omega_1 = \Omega_r + \frac{1-\pi}{2\pi-1}\varepsilon$ and $\Omega_2 = \Omega_r + \frac{\pi}{2\pi-1}\varepsilon$, where $\varepsilon = \frac{1}{\pi}\left(\frac{2\pi-1}{1-\pi}\frac{c}{2p-1} - f\right)$. Define function $\tilde{W}(\varepsilon) \equiv \hat{W}\left(\Omega_r + \frac{1-\pi}{2\pi-1}\varepsilon, \Omega_r + \frac{\pi}{2\pi-1}\varepsilon\right)$ for $\varepsilon \geq 0$. Differentiating,

$$
\tilde{W}'(\varepsilon) = \frac{\pi(1-\pi)}{2\pi-1}\left(\tilde{f}'(\Omega_r + \frac{1-\pi}{2\pi-1}\varepsilon) - \tilde{f}'(\Omega_r + \frac{\pi}{2\pi-1}\varepsilon)\right) = -\frac{\pi(1-\pi)}{2\pi-1}\int_{\Omega_r+\frac{1-\pi}{2\pi-1}}^{\Omega_r+\frac{\pi}{2\pi-1}\varepsilon} \tilde{f}''(x)\, dx.
$$

Therefore, a sufficient condition for $\tilde{W}(0) < \tilde{W}\left(\frac{1}{\pi}\left(\frac{2\pi-1}{1-\pi}\frac{c}{2p-1} - f\right)\right)$ is that $\tilde{f}''(x) < 0\ \forall x$, i.e., function $\tilde{f}(x)$ is concave. This result is established in Auxiliary Lemma A2.

Therefore, we can conclude that when multiple equilibria exist, i.e., when $f \in (\underline{f}, \bar{f})$, they rank in shareholder welfare in the following way: The equilibrium with incomplete crowding out of private information acquisition and $q_r < (2p-1) q_s$ has the highest shareholder welfare, followed by the equilibrium with incomplete crowding out of private information acquisition and $q_r > (2p-1) q_s$, which is followed by the equilibrium with complete crowding out of private information acquisition.

Proof of Proposition 3. The proposition directly follows from the welfare ranking in Lemma 2 and Assumption 2.

Proof of Proposition 4.

Using (27), the expected value from the decision is given by

$$
U = \sum_{k=\frac{N+1}{2}}^{N} \left(\pi P\left(p_a, N, k\right) + (1-\pi) P\left(p_d, N, k\right)\right) - \frac{1}{2}, \tag{32}
$$

where $p_a \equiv \Pr[v_i = 1|\theta = r = 1]$ and $p_d \equiv \Pr[v_i = 1|\theta = 1, r = 0]$, i.e., the equilibrium probability that a shareholder votes for the proposal given that it is beneficial ($\theta = 1$) and the proxy advisor's recommendation is correct and incorrect, respectively.

Proof of part 1. Note that the probability of a shareholder being pivotal in equilibrium with incomplete crowding out is the same as in the benchmark case:

$$\pi P(p_a, N-1, \frac{N-1}{2}) + (1-\pi) P(p_d, N-1, \frac{N-1}{2}) = \pi\Omega_1 + (1-\pi)\Omega_2 = \frac{2c}{2p-1}.$$

Consider the following optimization problem:

$$\begin{array}{c} \max_{p_a,p_d} \sum_{k=\frac{N+1}{2}}^{N} \left(\pi P(p_a, N, k) + (1-\pi) P(p_d, N, k)\right) \\ \text{s.to } \pi P\left(p_a, N-1, \frac{N-1}{2}\right) + (1-\pi) P\left(p_d, N-1, \frac{N-1}{2}\right) = \frac{2c}{2p-1} \end{array} \tag{33}$$

In what follows, we show that this optimization problem is solved by $p_a = p_d = \frac{1}{2} + q_0^* \left(p - \frac{1}{2}\right)$, i.e., the same as in the model without the proxy advisor. Let $x_a \equiv P(p_a, N-1, \frac{N-1}{2})$ and $x_d \equiv P(p_d, N-1, \frac{N-1}{2})$, and write the equivalent optimization problem as:

$$\begin{array}{c} \max_{x_a,x_d} \sum_{k=\frac{N+1}{2}}^{N} \left(\pi P(\varphi(x_a), N, k) + (1-\pi) P(\varphi(x_d), N, k)\right) \\ \text{s.t. } \pi x_a + (1-\pi) x_d = \frac{2c}{2p-1}, \end{array} \tag{34}$$

where $\varphi(x) \in (\frac{1}{2}, 1)$ is defined by (29). Auxiliary Lemma A1 at the end of the Appendix shows that the function $f(x) \equiv \sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k)$ is concave in x. Thus, by Jensen's inequality, for any x_a, x_d such that $\pi x_a + (1-\pi) x_d = \frac{2c}{2p-1}$, we have

$$\pi f(x_a) + (1-\pi) f(x_d) < f(\pi x_a + (1-\pi) x_d) = f(\frac{2c}{2p-1}) = \pi f(\frac{2c}{2p-1}) + (1-\pi) f(\frac{2c}{2p-1}).$$

Therefore, there is a unique solution to the maximization problem (33), given by $P(p_a, N-1, \frac{N-1}{2}) = P(p_d, N-1, \frac{N-1}{2}) = \frac{2c}{2p-1}$, which corresponds to the benchmark case. Hence, the efficiency of decision-making strictly declines compared to the benchmark case.

Proof of part 2. Next, we prove the second part of the proposition. In the equilibrium with complete crowding out of private information acquisition, we have

$$p_a = \tfrac{1}{2} + \tfrac{1}{2} q_r = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \left(\frac{f}{(2\pi-1) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}},$$
$$p_d = \tfrac{1}{2} - \tfrac{1}{2} q_r = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \left(\frac{f}{(2\pi-1) C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}.$$

Since $p_d = 1 - p_a$, we can re-write firm value as

$$U = \pi \sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k) + (1-\pi) \sum_{k=0}^{\frac{N-1}{2}} P(p_a, N, k) - \frac{1}{2} = \frac{1}{2} - \pi + (2\pi - 1) \sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k).$$

In contrast, the expected value from the decision in the benchmark case without the advisor is given by

$$U=\sum_{k=\frac{N+1}{2}}^{N}\left(P\left(p^{*},N,k\right)\right)-\frac{1}{2},$$

where

$$p^{*}=\frac{1}{2}+q_{0}^{*}\left(p-\frac{1}{2}\right)=\frac{1}{2}+\Omega=\frac{1}{2}+\sqrt{\frac{1}{4}-\left(C_{N-1}^{\frac{N-1}{2}}\frac{2p-1}{c}\right)^{-\frac{2}{N-1}}}.$$

It is higher with proxy advisor than without it if and only if

$$(2\pi-1)\sum_{k=\frac{N+1}{2}}^{N}P\left(p_{a},N,k\right)-\pi>\sum_{k=\frac{N+1}{2}}^{N}\left(P\left(p^{*},N,k\right)\right)-1.$$

Let us fix fee f and vary π. This equilibrium exists if and only if $f\leq\frac{2\pi-1}{2p-1}c$, i.e., $\pi\geq\frac{1}{2}+\frac{f}{c}\left(p-\frac{1}{2}\right)$. The derivative of the left-hand side in π is:

$$2\sum_{k=\frac{N+1}{2}}^{N}P\left(p_{a},N,k\right)+(2\pi-1)\frac{dp_{a}}{d\pi}\sum_{k=\frac{N+1}{2}}^{N}P_{q}\left(p_{a},N,k\right)-1>0,$$

since $\sum_{k=\frac{N+1}{2}}^{N}P\left(p_{a},N,k\right)>\frac{1}{2}$ and $\frac{dp_{a}}{d\pi}>0:$

$$\frac{dp_{a}}{d\pi}=\frac{1}{2\sqrt{\frac{1}{4}-\left(\frac{f}{(2\pi-1)C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}}\frac{2}{N-1}\left(\frac{f}{(2\pi-1)\,C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}-1}\frac{f}{C_{N-1}^{\frac{N-1}{2}}\,(2\pi-1)^{2}}>0.$$

Therefore, the left-hand side is strictly increasing in π.

Clearly, the advisor makes things worse for $\pi\to\frac{1}{2}+\frac{f}{c}\left(p-\frac{1}{2}\right)$. Indeed, in this case, $p_{a}\to p^{*}$, so we obtain

$$(2\pi-1)\sum_{k=\frac{N+1}{2}}^{N}P\left(p^{*},N,k\right)-\pi<\sum_{k=\frac{N+1}{2}}^{N}P\left(p^{*},N,k\right)-1\Leftrightarrow 1<2\sum_{k=\frac{N+1}{2}}^{N}P\left(p^{*},N,k\right),$$

which is true since $p^{*}>\frac{1}{2}$. When $\pi\to 1$, we have

$$p_{a}\to\frac{1}{2}+\sqrt{\frac{1}{4}-\left(\frac{f}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}>\frac{1}{2}+\sqrt{\frac{1}{4}-\left(C_{N-1}^{\frac{N-1}{2}}\frac{2p-1}{c}\right)^{-\frac{2}{N-1}}}=p^{*},$$

so the left-hand side converges to

$$\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2}+\sqrt{\frac{1}{4}-\left(\frac{f}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}, N, k\right)-1>\sum_{k=\frac{N+1}{2}}^{N} P\left(p^{*}, N, k\right)-1.$$

By monotonicity, there exists a unique $\pi^*(f) \in (\frac{1}{2}+\frac{f}{c}(p-\frac{1}{2}), 1)$ at which firm value is the same with the advisor as without.

Proof of Proposition 5. Consider the first statement of the proposition. The first part of Proposition 4 implies that if equilibrium features incomplete crowding out, then firm value is strictly lower than in the benchmark case. Hence, to find the conditions under which firm value is higher with the advisor, it is sufficient to find conditions under which the advisor sets fee in a way that crowds out private information acquisition. In case of complete crowding out, there is a one-to-one correspondence between the fee f set by the advisor and the fraction $q_r^H(f)$ buying its recommendation. Moreover, recall that the value of the advisor's signal to a shareholder is given by $V_r(q_r, 0) = (\pi - \frac{1}{2})P(\frac{1+q_r}{2}, N-1, \frac{N-1}{2})$ and must be equal to f. Thus, in this case, the advisor's problem is equivalent to maximizing $q_r V_r(q_r, 0)$ over q_r. Hence, instead of choosing fee f and maximizing $f q_r^H(f)$, the advisor can choose q_r and maximize $\eta(q_r) = P(\frac{1+q_r}{2}, N-1, \frac{N-1}{2})q_r$. Note that $\eta(q)$ is inverted U-shaped in q. Indeed,

$$P\left(\frac{1+q}{2}, N-1, \frac{N-1}{2}\right) q = C_{N-1}^{\frac{N-1}{2}}\left(\frac{(1+q)(1-q)}{4}\right)^{\frac{N-1}{2}} q = const \times q\left(1-q^2\right)^{\frac{N-1}{2}}$$

Differentiating the function of q,

$$\begin{aligned}
\left(1-q^2\right)^{\frac{N-1}{2}} - q\frac{N-1}{2}\left(1-q^2\right)^{\frac{N-1}{2}-1} 2q &= \left(1-q^2\right)^{\frac{N-1}{2}} - (N-1) q^2\left(1-q^2\right)^{\frac{N-1}{2}-1} \\
= \left(1-q^2\right)^{\frac{N-3}{2}}\left(1-q^2-(N-1) q^2\right) &= \left(1-q^2\right)^{\frac{N-3}{2}}\left(1-N q^2\right)
\end{aligned}$$

Hence, $\eta(q)$ is inverted U-shaped in q with a maximum at $q_m = \frac{1}{\sqrt{N}}$. The optimal fraction $q_m = \frac{1}{\sqrt{N}}$ translates into the optimal fee

$$f_m = (\pi - \frac{1}{2})P(\frac{1}{2}+\frac{1}{2\sqrt{N}}, N-1, \frac{N-1}{2}).$$

The fact that $\eta(q)$ is inverse U-shaped in q implies that under complete crowding out, the advisor's revenue is maximized at $f = f_m$ and is monotonically decreasing as f gets farther from f_m in both directions. Hence, the optimal pricing strategy of the advisor if $f_m > \underline{f}$ is to either set $f = \underline{f} - \varepsilon$, $\varepsilon \to 0$, or to choose the fee that maximizes its revenue under incomplete crowding out. In the second case, firm value is lower than in the benchmark case. In the first case, firm value converges to firm value with $f = \underline{f}$, which features incomplete crowding out and is shown to have lower firm value than in the benchmark case. Therefore, the only case where firm value can be higher than in the benchmark case is when $f_m < \underline{f}$. The constraint $f_m < \underline{f}$ can be simplified to

$$\pi > \hat{\pi} \equiv \frac{1}{2}\left(1 + \frac{C_{N-1}^{\frac{N-1}{2}} 2^{1-N} - \frac{2c}{2p-1}}{C_{N-1}^{\frac{N-1}{2}} 2^{1-N}\left(1 - \left(\frac{N-1}{N}\right)^{\frac{N-1}{2}}\right)}\right).$$

If each shareholder acquires the advisor's signal with probability q_r and remains uninformed otherwise, expected firm value is given by

$$\begin{aligned} V^*(\pi) = \Pr(\theta = 1) \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left[\pi P\left(q_r + \frac{1-q_r}{2}, N, k\right) + (1-\pi) P\left(\frac{1-q_r}{2}, N, k\right)\right] \\ - \Pr(\theta = 0) \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left[\pi P\left(\frac{1-q_r}{2}, N, k\right) + (1-\pi) P\left(q_r + \frac{1-q_r}{2}, N, k\right)\right] \\ = \tfrac{1}{2} \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left[(2\pi - 1) P\left(q_r + \frac{1-q_r}{2}, N, k\right) + (1-2\pi) P\left(\frac{1-q_r}{2}, N, k\right)\right] \\ = \left(\pi - \tfrac{1}{2}\right) \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left[P\left(\frac{1+q_r}{2}, N, k\right) - P\left(\frac{1-q_r}{2}, N, k\right)\right] = \\ \left(\pi - \tfrac{1}{2}\right) \textstyle\sum_{k=\frac{N+1}{2}}^{N} \left[P\left(\frac{1+q_r}{2}, N, k\right) - P\left(\frac{1+q_r}{2}, N, N-k\right)\right] \\ = \left(\pi - \tfrac{1}{2}\right)\left[2 \textstyle\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1+q_r}{2}, N, k\right) - 1\right] = (2\pi - 1)\left[\textstyle\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k\right) - \frac{1}{2}\right]. \end{aligned} \tag{35}$$

Comparing it with V_0, we get

$$\begin{aligned} (2\pi - 1)\left[\textstyle\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k\right) - \frac{1}{2}\right] > V_0 = \textstyle\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \Lambda, N, k\right) - \frac{1}{2} = \pi^* - \frac{1}{2} \\ \Leftrightarrow \pi > \tilde{\pi} \equiv \tfrac{1}{2} + \frac{\pi^* - \frac{1}{2}}{2\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k\right) - 1}. \end{aligned}$$

It can be shown that $\hat{\pi} < \tilde{\pi}$, and hence the presence of the advisor increases firm value if and only if $\pi > \tilde{\pi}$.

It remains to prove the second part of the proposition. Using (16), $\tilde{\pi}$ exceeds one if and only if

$$\frac{1}{2} + \frac{\pi^* - \frac{1}{2}}{2\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k\right) - 1} > 1 \Leftrightarrow \pi^* > \sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k\right).$$

By definition, $\pi^* = \sum_{k=\frac{N+1}{2}}^{N} P(p_0, N, k)$, where $p_0 \equiv pq_0^* + \frac{1-q_0^*}{2}$. Therefore, this inequality is equivalent to $p_0 > \frac{1}{2} + \frac{1}{2\sqrt{N}}$. Simplifying, we get $(2p-1)\, q_0^* > \frac{1}{\sqrt{N}}$.

Proof of Proposition 6. First, suppose that complete crowding out of private information acquisition occurs in equilibrium. According to (35), the expected value of the proposal is

$$(2\pi - 1)\left(\sum_{k=\frac{N+1}{2}}^{N} P\left(\frac{1+q_r}{2}, N, k\right) - \frac{1}{2}\right),$$

where $q_r = q_r^H(f)$ is given by (12). A marginal decrease in f increases q_r, which increases the expected value of the proposal. The case of incomplete crowding out of private information acquisition follows from the proof of Proposition 4: A marginal decrease in f increases the distance between x_a

and x_d, while keeping the total probability of being pivotal $(\pi x_a + (1-\pi) x_d)$ unchanged at $\frac{c}{2p-1}$. By concavity of function $f(x) \equiv \sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k)$, established in Auxiliary Lemma A1, this lowers the expected value of the proposal.

Proof of Proposition 7. Note that π^* is the equilibrium probability of making a correct decision in the benchmark model without the advisor.

1. First, consider $\pi \leq \pi^*$. If the fee satisfies $f \geq \bar{f}$, Lemma 2 implies that shareholders do not buy the advisor's recommendation, and hence firm value in the same as V_0, firm value in the benchmark case without the advisor. For any fee that does not deter shareholders from buying the advisor's recommendation ($f < \bar{f}$), we have two possible cases. If there is incomplete crowding out of private information acquisition, Proposition 4 shows that firm value is strictly lower than V_0. If there is complete crowding out of private information acquisition, the equilibrium probability of making a correct decision is strictly lower than π (because not all shareholders buy the advisor's recommendation – some remain uninformed), which in turn is lower than π^*. Since π^* is the equilibrium probability of making a correct decision in the benchmark case, firm value is again strictly lower than V_0. Thus, in both cases, setting $f \geq \bar{f}$ and deterring shareholders from buying the advisor's recommendation leads to a strictly higher firm value.

2. Second, consider $\pi > \pi^*$. If the fee is above $\bar{f}$ and hence $q_r = 0$, then firm value is exactly V_0. If the fee is such that $q_r > 0$ and there is incomplete crowding out of private information acquisition, Proposition 4 implies that firm value is strictly lower than V_0, and hence firm value could be increased by setting $f \geq \bar{f}$. Thus, such fee cannot be optimal. Finally, if the fee is such that $q_r > 0$ and there is complete crowding out of private information acquisition, (12) implies that the fraction of shareholders buying the advisor's recommendation monotonically decreases in the fee, so to maximize the number of informed shareholders and thereby firm value, it would be optimal to set the fee as low as possible in this range. As f converges to zero, (12) implies that q_r converges to one, i.e., all shareholders buy the advisor's recommendation. Hence, the probability of making a correct decision converges to π, which is strictly higher than π^*, the probability of making a correct decision in the benchmark case. Thus, indeed, the fee that maximizes firm value is arbitrarily close to zero.

Proof of Proposition 8. We first show that if the precision of the advisor's signal is not disclosed, the equilibrium of the game is the same as in the basic model but where the precision of the advisor's signal is the expected value of π, $\bar{\pi} \equiv \mu_l \pi_l + \mu_h \pi_h$. Indeed, fix the equilibrium probabilities q_r and q_s with which each shareholder acquires the advisor's signal and his private signal, and consider the information acquisition decision of any shareholder, taking the strategies of other shareholders as given. Denote $V_s(q_r, q_s, \pi)$ and $V_r(q_r, q_s, \pi)$ the shareholder's values from acquiring the private and public signal, respectively, if the precision of the advisor's signal is known to be π. These values are given by expressions (9) and (10). Then, the values from acquiring the private and public signal if the shareholder does not know the realization of π are $\bar{V}_s \equiv \mu_l V_s(q_r, q_s, \pi_l) + \mu_h V_s(q_r, q_s, \pi_h)$ and $\bar{V}_r \equiv \mu_l V_r(q_r, q_s, \pi_l) + \mu_h V_r(q_r, q_s, \pi_h)$. Because, $\Omega_1(q_r, q_s)$ and $\Omega_2(q_r, q_s)$ do not depend on π, (9) and (10) imply that $V_s(q_r, q_s, \pi)$ and $V_r(q_r, q_s, \pi)$ are linear in π. Hence, $\bar{V}_s = V_s(q_r, q_s, \bar{\pi})$ and $\bar{V}_r = V_r(q_r, q_s, \bar{\pi})$. This proves that the equilibrium of the game without disclosure coincides with the equilibrium of the basic model with precision $\bar{\pi}$.

Denote $V^*(\pi)$ the expected value of the proposal in the equilibrium of the basic model when the precision of the advisor's signal is π. The argument above implies that the expected value of the proposal in the game without disclosure is given by $V^*(\bar{\pi})$. Since the expected value of the proposal in the game with disclosure is $\mu_l V^*(\frac{1}{2}) + \mu_h V^*(\pi_h)$ and since $V^*(\frac{1}{2}) = V_0$, given by (7), we want to prove that under each of the conditions of the proposition, $\mu_l V_0 + \mu_h V^*(\pi_h) > V^*(\bar{\pi})$.

Consider the first statement of the proposition, i.e., suppose that $V^*(\pi_h) > V_0$. First, if $\bar{\pi}$ is such that $V^*(\bar{\pi}) \leq V_0$, we have $\mu_l V_0 + \mu_h V^*(\pi_h) > V_0 \geq V^*(\bar{\pi})$, as required. Second, consider $\bar{\pi}$ such that $V^*(\bar{\pi}) > V_0$. The proof of Proposition 5 implies that $\bar{\pi} \geq \tilde{\pi}$, $f^* = f_m$, and hence $V^*(\bar{\pi})$ is given by (15). Since $V^*(\pi_h) > V_0$, $V^*(\pi_h)$ is also given by (15). Hence,

$$\begin{gathered} V^*(\bar{\pi}) = (2\bar{\pi} - 1)\left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k) - \frac{1}{2}\right) \\ = \mu_h (2\pi_h - 1)\left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P(\frac{1}{2} + \frac{1}{2\sqrt{N}}, N, k) - \frac{1}{2}\right) = \mu_h V^*(\pi_h) < \mu_l V_0 + \mu_h V^*(\pi_h), \end{gathered}$$

as required.

Next, consider the second statement of the proposition. If $V^*(\pi_h) > V_0$, then the first statement of the proposition, which has just been proved, applies. Hence, consider $V^*(\pi_h) \leq V_0$. Note that in the range of complete crowding out of private information acquisition, the quality of decision-making $V^*(\pi)$ is strictly increasing in π. Therefore, $V^*(\pi_h) > V^*(\bar{\pi})$. Hence, $\mu_l V_0 + \mu_h V^*(\pi_h) \geq V^*(\pi_h) > V^*(\bar{\pi})$, as required.

Auxiliary Lemma A1. *Function $f(x) \equiv \sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k)$, where $\varphi(x)$ is defined by (29), is concave.*

Proof of Auxiliary Lemma A1. It will be useful to compute the derivative:

$$\varphi'(x) = -\frac{1}{C_{N-1}^{\frac{N-1}{2}} (N-1)\, \psi(x)}, \tag{36}$$

where

$$\psi(x) \equiv \left(\frac{x}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{N-3}{N-1}} \sqrt{\frac{1}{4} - \left(\frac{x}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}}.$$

Note that

$$\begin{gathered} f''(x) = \left(\tfrac{d\varphi}{dx}\right)^2 \left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P_{qq}(\varphi(x), N, k)\right) + \tfrac{d^2\varphi}{dx^2}\left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P_q(\varphi(x), N, k)\right) \\ = \frac{1}{\left(C_{N-1}^{\frac{N-1}{2}}\right)^2 (N-1)^2 \psi(x)^2}\left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P_{qq}(\varphi(x), N, k)\right) + \frac{\psi'(x)}{C_{N-1}^{\frac{N-1}{2}} (N-1)\psi(x)^2}\left(\textstyle\sum_{k=\frac{N+1}{2}}^{N} P_q(\varphi(x), N, k)\right) \end{gathered}$$

Simplifying,

$$\begin{gathered} \left(C_{N-1}^{\frac{N-1}{2}}\right)^2 (N-1)^2 \psi(x)^2 f''(x) \\ = \textstyle\sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k)\left(\left(\frac{k - N\varphi(x)}{\varphi(x)(1-\varphi(x))}\right)^2 - \frac{k}{\varphi(x)^2} - \frac{N-k}{(1-\varphi(x))^2} + C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi'(x)\left(\frac{k-N\varphi(x)}{\varphi(x)(1-\varphi(x))}\right)\right). \end{gathered}$$

Next, we can calculate $\psi'(x)$:

$$
\begin{aligned}
C_{N-1}^{\frac{N-1}{2}}(N-1)\psi'(x) &= \left(\frac{N-3}{4}\left(\frac{x}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{-2}{N-1}} - N + 2\right)\left(\frac{1}{4} - \left(\frac{x}{C_{N-1}^{\frac{N-1}{2}}}\right)^{\frac{2}{N-1}}\right)^{-\frac{1}{2}} \\
&= \frac{1}{\varphi(x)-\frac{1}{2}}\left(\frac{N-3}{4}\frac{1}{\varphi(x)(1-\varphi(x))} - N + 2\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left(C_{N-1}^{\frac{N-1}{2}}\right)^2 (N-1)^2 \psi(x)^2 f''(x) \\
&= \sum_{k=\frac{N+1}{2}}^{N} P(\varphi(x), N, k)\left(\begin{array}{c} \left(\frac{k-N\varphi(x)}{\varphi(x)(1-\varphi(x))}\right)^2 - \frac{k}{\varphi(x)^2} - \frac{N-k}{(1-\varphi(x))^2} \\ +\frac{1}{\varphi(x)-\frac{1}{2}}\left(\frac{N-3}{4}\frac{1}{\varphi(x)(1-\varphi(x))} - N + 2\right)\left(\frac{k-N\varphi(x)}{\varphi(x)(1-\varphi(x))}\right) \end{array}\right).
\end{aligned}
$$

Multiplying by $(\varphi(x)(1-\varphi(x)))^2$:

$$
\begin{aligned}
&\left(C_{N-1}^{\frac{N-1}{2}}\right)^2 (N-1)^2 \psi(x)^2 (\varphi(x)(1-\varphi(x)))^2 f''(x) \\
&= \sum_{k=\frac{N+1}{2}}^{N} P(q, N, k)\left(\begin{array}{c} (k-Nq)^2 - k(1-q)^2 - (N-k)q^2 \\ +\frac{1}{q-\frac{1}{2}}\left(\frac{N-3}{4} - (N-2)q(1-q)\right)(k-Nq) \end{array}\right),
\end{aligned}
$$

where we denote $\varphi(x)$ by $q \in \left(\frac{1}{2}, 1\right)$. We want to show that this expression is negative. Since $\sum_{k=0}^{N} P_q(q, N, k) = 0$ and $\sum_{k=0}^{N} P_{qq}(q, N, k) = 0$,

$$
f''(x) = -\left(\frac{d\varphi}{dx}\right)^2\left(\sum_{k=0}^{\frac{N-1}{2}} P_{qq}(\varphi(x), N, k)\right) - \frac{d^2\varphi}{dx^2}\left(\sum_{k=0}^{\frac{N-1}{2}} P_q(\varphi(x), N, k)\right).
$$

Therefore $f''(x) < 0$ if the following expression is positive:

$$
L = \sum_{k=0}^{\frac{N-1}{2}} P(q, N, k)\left((k-Nq)^2 - k(1-q)^2 - (N-k)q^2 + \frac{2(k-Nq)}{2q-1}\left(\frac{N-3}{4} - (N-2)q(1-q)\right)\right)
$$

for any $q \in \left(\frac{1}{2}, 1\right)$. Let

$$
\zeta(q, k) \equiv (k-Nq)^2 - k(1-q)^2 - (N-k)q^2 + C(k-Nq),
$$

where $C \equiv \frac{2}{2q-1}\left(\frac{N-3}{4} - (N-2)q(1-q)\right)$. Hence,

$$
\zeta(q, k) = k(k-1) - (2(N-1)q - C)k + N(N-1)q^2 - CNq.
$$

Hence,

$$L=\sum_{k=0}^{\frac{N-1}{2}} P\left(q,N,k\right)k\left(k-1\right)-\left(2\left(N-1\right)q-C\right)\sum_{k=0}^{\frac{N-1}{2}} P\left(q,N,k\right)k+\left(N\left(N-1\right)q^{2}-CNq\right)\sum_{k=0}^{\frac{N-1}{2}} P\left(q,N,k\right).$$

Consider the first two terms:

1. Term 1:

$$\begin{aligned}\textstyle\sum_{k=0}^{\frac{N-1}{2}} k\left(k-1\right)C_N^k q^k\left(1-q\right)^{N-k}=\sum_{k=2}^{\frac{N-1}{2}} k\left(k-1\right)\frac{N!}{k!(N-k)!}q^k\left(1-q\right)^{N-k}\\ =N\left(N-1\right)q^2\textstyle\sum_{m=0}^{\frac{N-1}{2}-2} P\left(q,N-2,m\right)=N\left(N-1\right)q^2\Pr\left[k\le\frac{N-1}{2}-2|k\sim B\left(N-2,q\right)\right].\end{aligned}$$

2. Term 2:

$$\begin{aligned}\textstyle\sum_{k=0}^{\frac{N-1}{2}} kC_N^k q^k\left(1-q\right)^{N-k}=\sum_{k=1}^{\frac{N-1}{2}} k\frac{N!}{k!(N-k)!}q^k\left(1-q\right)^{N-k}\\ =qN\left(\textstyle\sum_{k=0}^{\frac{N-1}{2}-1} P\left(q,N-1,k\right)\right)=qN\Pr\left[k\le\frac{N-1}{2}-1|k\sim B\left(N-1,q\right)\right].\end{aligned}$$

Hence,

$$\begin{aligned}\tfrac{L}{qN}=\left(N-1\right)q\Pr\left[k\le\tfrac{N-1}{2}-2|k\sim B\left(N-2,q\right)\right]\\ -\left(2\left(N-1\right)q-C\right)\Pr\left[k\le\tfrac{N-1}{2}-1|k\sim B\left(N-1,q\right)\right]\\ +\left(\left(N-1\right)q-C\right)\Pr\left[k\le\tfrac{N-1}{2}|k\sim B\left(N,q\right)\right].\end{aligned}$$

Note that

$$\begin{aligned}\Pr\left[k\le\tfrac{N-1}{2}|k\sim B\left(N,q\right)\right]=I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N+1}{2}\right),\\ \Pr\left[k\le\tfrac{N-1}{2}-1|k\sim B\left(N-1,q\right)\right]=I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right),\\ \Pr\left[k\le\tfrac{N-1}{2}-2|k\sim B\left(N-2,q\right)\right]=I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-3}{2}\right),\end{aligned}$$

where $I_{1-q}\left(\cdot\right)$ is the regularized incomplete beta function. Using the properties of the regularized incomplete beta function,

$$\begin{aligned}I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N+1}{2}\right)=I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right)+\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-1}{2}}}{\frac{N-1}{2}B\left(\frac{N+1}{2},\frac{N-1}{2}\right)}\\ I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right)=I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-3}{2}\right)+\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-3}{2}}}{\frac{N-3}{2}B\left(\frac{N+1}{2},\frac{N-3}{2}\right)}.\end{aligned}$$

Plugging into the expression for $\frac{L}{qN}$:

$$\begin{aligned}\tfrac{L}{qN}=\left(N-1\right)q\left(I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right)-\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-3}{2}}}{\frac{N-3}{2}B\left(\frac{N+1}{2},\frac{N-3}{2}\right)}\right)-\left(2\left(N-1\right)q-C\right)I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right)\\ +\left(\left(N-1\right)q-C\right)\left(I_{1-q}\left(\tfrac{N+1}{2},\tfrac{N-1}{2}\right)+\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-1}{2}}}{\frac{N-1}{2}B\left(\frac{N+1}{2},\frac{N-1}{2}\right)}\right)\\ =-\left(N-1\right)q\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-3}{2}}}{\frac{N-3}{2}B\left(\frac{N+1}{2},\frac{N-3}{2}\right)}+\left(\left(N-1\right)q-C\right)\frac{(1-q)^{\frac{N+1}{2}}q^{\frac{N-1}{2}}}{\frac{N-1}{2}B\left(\frac{N+1}{2},\frac{N-1}{2}\right)}.\end{aligned}$$

Dividing by $(1-q)^{\frac{N+1}{2}} q^{\frac{N-3}{2}}$ and simplifying,

$$\frac{L}{(1-q)^{\frac{N+1}{2}} q^{\frac{N-1}{2}} N} = \frac{q\,(N-1)!}{\left(\frac{N-1}{2}\right)!\left(\frac{N-3}{2}\right)!}(2q-1) - C\frac{q\,(N-1)!}{\frac{N-1}{2}\left(\frac{N-1}{2}\right)!\left(\frac{N-3}{2}\right)!}.$$

Hence,

$$\begin{aligned} \frac{L\left(\frac{N-3}{2}\right)!\left(\frac{N-1}{2}\right)!(2q-1)}{(1-q)^{\frac{N+1}{2}} q^{\frac{N+1}{2}} N!} &= (2q-1)^2 - \tfrac{2}{N-1}\left(\tfrac{N-3}{2} - 2\,(N-2)\,q\,(1-q)\right) \\ = \tfrac{4}{N-1}q^2 - \tfrac{4}{N-1}q + \tfrac{2}{N-1} &\Leftrightarrow \frac{L\left(\frac{N-3}{2}\right)!\left(\frac{N-1}{2}\right)!(2q-1)(N-1)}{(1-q)^{\frac{N+1}{2}} q^{\frac{N+1}{2}} N!2} = 2q^2 - 2q + 1. \end{aligned} \tag{37}$$

Since $2q^2 - 2q + 1 > 0$, we conclude that $L > 0$ for any $q \in (\frac{1}{2}, 1)$. Therefore, $f''(x) < 0$, which completes the proof.

Auxiliary Lemma A2. *Function $\tilde{f}(x)$, defined by (31), is concave.*

Proof of Auxiliary Lemma A2. Differentiating $\tilde{f}(x)$ and using the definition of $f(x)$,

$$\tilde{f}''(x) = f''(x) - 2\varphi'(x) - x\varphi''(x).$$

Using $f''(x)$ from the proof of Auxiliary Lemma A1 above, in particular, expression (37), (36), and its derivative, we can write

$$\tilde{f}''(x) = x\frac{(2\varphi(x)^2 - 2\varphi(x) + 1)N}{(2\varphi(x)-1)\,\varphi(x)\,(1-\varphi(x))\,(C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi(x))^2} + \frac{2}{C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi(x)} - \frac{x\psi'(x)}{C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi(x)^2}$$

Multiplying both sides by $\left(C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi(x)\right)^2$, using

$$C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi'(x) = \frac{2}{2\varphi(x)-1}\left(\frac{N-3}{4}\frac{1}{\varphi(x)\,(1-\varphi(x))} - N + 2\right),$$

and simplifying, we obtain

$$\left(C_{N-1}^{\frac{N-1}{2}}(N-1)\,\psi(x)\right)^2 \tilde{f}''(x) = -\frac{(N-1)\,x}{(2\varphi(x)-1)\,\varphi(x)\,(1-\varphi(x))} < 0,$$

since $\varphi(x) \in \left(\frac{1}{2}, 1\right)$. Therefore, $\tilde{f}(x)$ is concave.